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REGULARITY THEORY FOR SOLUTIONS TO SECOND ORDER ELLIPTIC OPERATORS WITH COMPLEX COEFFICIENTS AND THE L^p DIRICHLET PROBLEM

MARTIN DINDOŠ AND JILL PIPHER

ABSTRACT. We establish a new theory of regularity for elliptic complex valued second order equations of the form $\mathcal{L} = \operatorname{div} A(\nabla \cdot)$, when the coefficients of the matrix A satisfy a natural algebraic condition, a strengthened version of a condition known in the literature as L^p -dissipativity. Precisely, the regularity result is a reverse Hölder condition for L^p averages of solutions on interior balls, and serves as a replacement for the De Giorgi - Nash - Moser regularity of solutions to real-valued divergence form elliptic operators. In a series of papers, Cialdea and Maz'ya studied necessary and sufficient conditions for L^p -dissipativity of second order complex coefficient operators and systems. Recently, Carbonaro and Dragičević introduced a condition they termed p -ellipticity, and showed that it had implications for boundedness of certain bilinear operators that arise from complex valued second order differential operators. Their p -ellipticity condition is exactly our strengthened version of L^p -dissipativity. The regularity results of the present paper are applied to solve L^p Dirichlet problems for $\mathcal{L} = \operatorname{div} A(\nabla \cdot) + B \cdot \nabla$ when A and B satisfy a Carleson measure condition, which previously was known only in the real valued case. We show solvability of the L^2 Dirichlet problem, as well as solvability of the L^p Dirichlet boundary value problem for p in the range where A is p -elliptic.

1. INTRODUCTION

In this paper, we establish a new theory of interior regularity of solutions to second order divergence form complex coefficient operators $\mathcal{L} = \operatorname{div} A(x) \nabla + B(x) \cdot \nabla$ under certain natural algebraic conditions on the matrix A and a natural minimal scaling condition on B , without any additional smoothness of the coefficients. If the coefficients of A and B are real, the algebraic conditions on A are precisely uniform ellipticity.

The improvements in regularity of solutions, as expressed as (1.10) and (1.11) of Theorem 1.1 below, can be used as a substitute for the De Giorgi-Nash-Moser regularity theory for real divergence form elliptic equations. In the latter case, we know that when A, B are real valued and A is elliptic, the regularity theory for solutions gives that $u \in C^\alpha(B)$; this need not hold for solutions to complex coefficient operators. Indeed, solutions need not be locally bounded. However, we show that an iterative procedure, reminiscent of Moser's iteration scheme, can be amplified in a range determined by these algebraic conditions, yielding greater regularity of weak solutions. Moreover, we apply this regularity to show solvability of a Dirichlet boundary value problem for a class of variable coefficient complex coefficient second order operators, with certain minimal and natural smoothness assumptions on the coefficients. The solvability of these boundary value problems in the complex case require new ideas which make the methods of [27] and [18]

more broadly applicable. Further, the regularity theory established in this paper should make possible the study of boundary value problems for a variety of complex coefficient operators.

We recently found that Carbonaro and Dragičević [7] have also formulated this same algebraic condition (which they termed p -ellipticity) and showed that it had implications for boundedness of certain bilinear operators that arise from complex valued second order differential operators. Their dimension-free bounds are related to the question of contractivity of the associated semigroup of these operators in L^p . The issue of L^p contractivity of semigroups had been considered in a series of papers by Cialdea and Maz'ya [8–10], and we next introduce and discuss this in more detail.

In the case of complex coefficients, the usual ellipticity assumption is that there exist constants $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\xi|^2 \leq \Re \sum_{i,j=0}^{n-1} A_{ij}(x) \xi_i \overline{\xi_j} = \Re \langle A(x)\xi, \xi \rangle \quad \text{and} \quad |\langle A\xi, \eta \rangle| \leq \Lambda|\xi||\eta| \quad (1.1)$$

for all $\xi, \eta \in \mathbb{C}^n$ and a.e. $x \in \Omega$. In this paper, we consider a stronger form of ellipticity, a strengthening of the concept of L^p dissipativity as defined in [8–10], which in turn was motivated by understanding when semigroups generated by second order elliptic operators are contractive in L^p . In particular, it had long been known that scalar second order elliptic operators with real coefficients generate contractive semigroups in L^p for all $1 \leq p \leq \infty$.

In [8], the following condition was shown to be sufficient for L^p dissipativity:

$$\frac{4}{pp'} \langle \Re A(x)\xi, \xi \rangle + \langle \Re A(x)\eta, \eta \rangle + 2\langle (p^{-1} \mathcal{J}m A(x) - p'^{-1} \mathcal{J}m A^t(x))\xi, \eta \rangle \geq 0, \quad (1.2)$$

for all $\xi, \eta \in \mathbb{R}^n$.

We will consider a stronger condition, and use a change of variables $\xi = \frac{\sqrt{pp'}}{2}\lambda$ to write it as follows: For some $\varepsilon > 0$ and all $\lambda, \eta \in \mathbb{R}^n$

$$\langle \Re A \lambda, \lambda \rangle + \langle \Re A \eta, \eta \rangle + \left\langle \left(\sqrt{\frac{p'}{p}} \mathcal{J}m A - \sqrt{\frac{p}{p'}} \mathcal{J}m A^t \right) \lambda, \eta \right\rangle \geq \varepsilon(|\lambda|^2 + |\eta|^2). \quad (1.3)$$

This same condition also appears in the paper [7], where the authors introduce it in the following form. For $p > 1$ define the \mathbb{R} -linear map $\mathcal{J}_p : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$\mathcal{J}_p(\alpha + i\beta) = \frac{\alpha}{p} + i \frac{\beta}{p'}$$

where $p' = p/(p-1)$ and $\alpha, \beta \in \mathbb{R}^n$. They define the matrix A to be p -elliptic if for a.e. $x \in \Omega$

$$\Re \langle A(x)\xi, \mathcal{J}_p \xi \rangle \geq \lambda_p |\xi|^2, \quad \forall \xi \in \mathbb{C}^n \quad (1.4)$$

for some $\lambda_p > 0$.

Henceforth, A will be called p -elliptic if it satisfies (1.4) and the upper bound

$$|\langle A(x)\xi, \eta \rangle| \leq \Lambda |\xi| |\eta|, \quad \forall \xi, \eta \in \mathbb{C}^n. \quad (1.5)$$

A short calculation shows that (1.4) and (1.3) are equivalent.

We shall adopt the notation of [7], and recall their observation that this strengthened ellipticity condition is equivalent to $\Delta_p(A) > 0$ where

$$\Delta_p(A) = \operatorname{ess\,inf}_{x \in \Omega} \min_{|\xi|=1} \Re \langle A(x)\xi, \mathcal{J}_p \xi \rangle. \quad (1.6)$$

Observe that when $p = 2$ this is just the usual ellipticity condition (1.1). The p -ellipticity condition $\Delta_p(A) > 0$ can be restated in a different form

$$|1 - 2/p| < \mu(A), \quad (1.7)$$

where

$$\mu(A) = \operatorname{ess\,inf}_{(x,\xi) \in \Omega \times \mathbb{C}^n \setminus \{0\}} \Re \frac{\langle A(x)\xi, \xi \rangle}{|\langle A(x)\xi, \xi \rangle|}. \quad (1.8)$$

(c.f. Proposition 5.14 of [7]). The advantage of writing the inequality in this form is that it separates A from p . It also immediately implies that if a matrix A is elliptic (i.e., (1.1) holds) then there exists $p_0 \in [1, 2)$ such that A is p -elliptic if and only if $p \in (p_0, p'_0)$, where $p_0 = 2/(1 + \mu(A))$. Moreover, $p_0 = 1$ if and only if the matrix A is real and the quantity $\mu(A)$ is trivially bounded by $\mu(A) \geq \lambda/\Lambda$ giving a trivial upper bound on the value of p_0 .

Our first main result concerns solutions to $\mathcal{L} = \operatorname{div} A(x) \nabla + B(x) \cdot \nabla$ in a domain $\Omega \subset \mathbb{R}^n$.

Theorem 1.1. *Suppose that $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ is the weak solution to the operator $\mathcal{L}u := \operatorname{div} A(x) \nabla u + B(x) \cdot \nabla u = 0$ in Ω , an open subset of \mathbb{R}^n . Let $p_0 = \inf\{p > 1 : A \text{ is } p\text{-elliptic}\}$, and suppose that B has measurable coefficients $B_i \in L_{loc}^\infty(\Omega)$ satisfying the condition*

$$|B_i(x)| \leq K(\delta(x))^{-1}, \quad \forall x \in \Omega \quad (1.9)$$

where the constant K is uniform, and $\delta(x)$ denotes the distance of x to the boundary of Ω . Then we have the following improvement in the regularity of u . For any $B_{4r}(x) \subset \Omega$ and $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left(\int_{B_r(x)} |u|^p dy \right)^{1/p} \leq C_\varepsilon \left(\int_{B_{2r}(x)} |u|^q dy \right)^{1/q} + \varepsilon \left(\int_{B_{2r}(x)} |u|^2 dy \right)^{1/2} \quad (1.10)$$

for all $p, q \in (p_0, \frac{p'_0 n}{n-2})$. (Here $p'_0 = p_0/(p_0 - 1)$ and when $n = 2$ one can take $p, q \in (p_0, \infty)$.) The constant in the estimate depends on the dimension, the p -ellipticity constants, Λ , K and $\varepsilon > 0$ but not on $x \in \Omega$, $r > 0$ or u . Moreover, for all $p \in (p_0, p'_0)$ and any $\varepsilon > 0$

$$r^2 \int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \leq C_\varepsilon \int_{B_{2r}(x)} |u(y)|^p dy + \varepsilon \left(\int_{B_{2r}(x)} |u(y)|^2 dy \right)^{p/2}, \quad (1.11)$$

where the constant again depend only on the dimension, p , Λ , K and $\varepsilon > 0$. In particular, $|u|^{(p-2)/2} u$ belongs to $W_{loc}^{1,2}(\Omega; \mathbb{C})$.

Remark. Clearly, if $q \geq 2$ in (1.10) and if $p \geq 2$ in (1.11) one can take $\varepsilon = 0$ as the L^2 average of u can be controlled by the first term on the right hand side of each of the two inequalities.

In general, one can not expect a larger range of p in the reverse Hölder condition of (1.10). In [28], Mayboroda gives a counterexample to (1.10) when $q = 2$ and for any $p > \frac{2n}{n-2}$ under the assumption of (1.1) (which is the same as 2-ellipticity).

We apply this regularity result to the question of solvability of L^p Dirichlet problem for elliptic operators of this type. This part of the paper is motivated by the

known results concerning boundary value problems for second order elliptic equations in divergence form, when the coefficients are **real** and satisfy a certain natural, minimal smoothness condition (refer [18, 19, 27]). The literature on solvability of boundary value problems for complex coefficient operators in \mathbb{R}^n is limited, except when the matrix A is of block form. For block matrices A in $\mathcal{L} = \operatorname{div} A(x) \nabla$, there are numerous results on L^p -solvability of the Dirichlet, regularity and Neumann problems, starting with the solution of the Kato problem, where the coefficients of the block matrix are also assumed to be independent of the transverse variable (this assumption is usually referred in literature as “ t -independent”, in our notation it is the x_0 variable). See [6] and [23] and the references therein. For matrices not of block form, there are solvability results in various special cases assuming that the solutions satisfy De Giorgi - Nash - Moser estimates. See [1] and [22] for example. The latter paper is also concerned with operators that are t -independent. Finally, there are perturbation results in a variety of special cases, such as [3] and [2]; the first paper shows that solvability in L^2 implies solvability in L^p for p near 2, and the second paper has L^2 -solvability results for small L^∞ perturbations of real elliptic operators when the complex matrix is t -independent.

Our solvability result for operators of the form $\mathcal{L} = \operatorname{div} A(x) \nabla + B(x) \cdot \nabla$ can be applied on a domains above a Lipschitz graph in \mathbb{R}^n . We do not assume “ t -independence”. Instead, we assume the coefficients A and B satisfy a natural Carleson condition that has appeared in the literature so far only for real elliptic operators. ([27], [18], and [19]). The Carleson condition on A , (1.12) below, holds uniformly on Lipschitz subdomains, and is thus a natural condition in the context of chord-arc domains as well. The paper [24] connects geometric information about the boundary of the domain to information about the elliptic measure of operators that satisfy some closely related conditions.

The second main theorem of the paper establishes the solvability of $\mathcal{L}u = 0$ with L^p Dirichlet boundary data for variable coefficient complex coefficient operators satisfying these Carleson conditions on coefficients. The solvability of the L^2 Dirichlet problem (where 2-ellipticity is the standard assumption) for complex coefficient operators satisfying (1.12) is a consequence of the L^2 results for non-symmetric elliptic systems in [17]. In fact, as is typical in this theory, Dindoš, Hwang, and M. Mitrea actually obtain in [17] L^p results for elliptic systems, for $p \in (2 - \varepsilon, 2 + \varepsilon)$. One of the novelties of this paper is showing that p -ellipticity extends solvability of the L^p -Dirichlet problem to a broader range of L^p . To prove this, we adapt the method of proof that was established in [27]; however, the lack of continuity of solutions and the absence of a maximum principle requires new ideas to generalize this approach. It turns out the p -ellipticity condition is also the correct one for establishing solvability of perturbations of elliptic operators as well as solvability of the regularity problems. We shall take this up in a separate manuscript.

Theorem 1.2. *Let $1 < p < \infty$, and let Ω be the upper half-space $\mathbb{R}_+^n = \{(x_0, x') : x_0 > 0 \text{ and } x' \in \mathbb{R}^{n-1}\}$. Consider the operator*

$$\mathcal{L}u = \partial_i (A_{ij}(x) \partial_j u) + B_i(x) \partial_i u$$

and assume that the matrix A is p -elliptic with constants λ_p, Λ , $A_{00} = 1$ and $\mathcal{I}m A_{0j} = 0$ for all $1 \leq j \leq n-1$. Assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} [|\nabla A|^2 + |B|^2] \delta(x) dx \quad (1.12)$$

is a Carleson measure in Ω . Let us also denote

$$d\mu'(x) = \sup_{B_{\delta(x)/2}(x)} \left[\sum_j |\partial_0 A_{0j}|^2 + \left| \sum_j \partial_j A_{0j} \right|^2 + |B|^2 \right] \delta(x) dx. \quad (1.13)$$

Then there exist $K = K(\lambda_p, \Lambda, \|\mu\|_{\mathcal{C}}, n, p) > 0$ and $C(\lambda_p, \Lambda, \|\mu\|_{\mathcal{C}}, n, p) > 0$ such that if

$$\|\mu'\|_{\mathcal{C}} < K \quad (1.14)$$

then the L^p -Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = f & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\ \tilde{N}_{p,a}(u) \in L^p(\partial\Omega), \end{cases} \quad (1.15)$$

is solvable and the estimate

$$\|\tilde{N}_{p,a}(u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{C})} \quad (1.16)$$

holds for all energy solutions u with datum f .

In the statement of this theorem, we've used some notation that will be defined in subsequent sections. We will also recall there the concept of Carleson measure, discuss the notions of L^p solvability and energy solutions and define \tilde{N}_p which is a variant of the nontangential maximal function defined using L^p averages of the solution u . By Theorem 1.1, instead of L^p averages we could use L^q averages for q in the range $(p_0, \frac{p_0 n}{n-2})$ to obtain the same result. Also, see section 4 for a detailed discussion of some further assumptions we make to prove Theorem 1.2 on Lipschitz domains.

Classically, the L^p boundedness of the nontangential maximal function of a solution (in our case, estimate (1.16) above) gives nontangential convergence of the solution to its boundary values. Since the nontangential maximal function of our complex-valued solution will require smoothing by averaging, we will also get a nontangential convergence result, but stated for averages of solutions. This convergence of averages is a consequence of solvability in L^p and is not connected with the assumptions on the coefficients of the equation. For this reason, that very general result is given in an appendix at the end.

We now state, as a further corollary of the second main theorem, a result for matrices A in "block form". This corollary uses the fact that the assumption that the Carleson measure norm is small is needed only on the last row (A_{0j}) of the matrix A . This latter observation was pointed out to us by S. Mayboroda - see [15] and also [14]. Hence, we have the following.

Corollary 1.3. *Suppose the operator \mathcal{L} on \mathbb{R}_+^n has the form*

$$\mathcal{L}u = \partial_0^2 u + \sum_{i,j=1}^{n-1} \partial_i (A_{ij} \partial_j u)$$

where the matrix A has coefficients satisfying the Carleson condition (1.12).

Then for all $1 < p < \infty$ for which A is p -elliptic, the L^p -Dirichlet problem (4.1) is solvable for \mathcal{L} and the estimate

$$\|\tilde{N}_{p,a}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega;\mathbb{C})} \quad (1.17)$$

holds for all energy solutions u with datum f .

The paper is organized as follows. In Section 2, we discuss the p -ellipticity condition and prove interior regularity of solutions to complex valued divergence form equations under this assumption. Section 3 contains preliminary material relevant to defining solvability of the Dirichlet problem for second order divergence form operators whose coefficients satisfy a Carleson measure condition. Section 4 introduces the L^p -Dirichlet problem in an infinite strip in order to deal with issues around finiteness of the classical nontangential maximal functions associated with solvability of this problem. The estimates for an appropriate square function (the p -adapted square function of [18]) can be found in Section 5: the integration by parts argument here uses p -ellipticity and the Carleson measure condition on the coefficients in a crucial way. The estimates for the nontangential maximal function (in terms of the square function) in Section 6 rely on a strategy introduced in [25] and extended in [17] to handle elliptic systems in L^2 . In the Appendix, we address the issue of nontangential convergence of solutions.

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2. L^p DISSIPATIVITY, p -ELLIPTICITY AND REGULARITY RESULTS

The concept of L^p dissipativity was defined in a series of papers by Cialdea and Maz'ya [8–10] and was motivated by the effort to characterize when semigroups generated by second order elliptic operators are contractive in L^p . In particular, it has long been known that scalar second order elliptic operators with real coefficients generate contractive semigroups in L^p for all $1 \leq p \leq \infty$. The case of operators generating L^∞ -contractive semigroups is studied in [4].

Following [8] let $\mathcal{L}(u, v)$ be the sesquilinear form

$$\mathcal{L}(u, v) = \int_{\Omega} \langle A(x) \nabla u, \nabla v \rangle dx,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on \mathbb{C}^n . Clearly, $\mathcal{L}(u, v)$ is well defined for $(u, v) \in C_0^1(\Omega) \times C_0^1(\Omega)$ consisting of complex value functions having compact support in Ω with continuous first derivative.

Definition 2.1. (Cialdea-Maz'ya) Let $1 < p < \infty$. The form \mathcal{L} is called L^p dissipative if for all $u \in C_0^1(\Omega)$

$$\operatorname{Re} \mathcal{L}(u, |u|^{p-2}u) \geq 0, \quad \text{if } p \geq 2, \quad (2.1)$$

$$\operatorname{Re} \mathcal{L}(u|u|^{p'-2}, u) \geq 0, \quad \text{if } 1 < p < 2. \quad (2.2)$$

Theorem 2.2. [8, Theorem 1, Corollary 4 and Corollary 6] A sufficient condition for the form \mathcal{L} to be L^p dissipative is that

$$\frac{4}{pp'} \langle \Re A(x) \xi, \xi \rangle + \langle \Re A(x) \eta, \eta \rangle + 2 \langle (p^{-1} \mathcal{I} m A(x) - p'^{-1} \mathcal{I} m A^t(x)) \xi, \eta \rangle \geq 0, \quad (2.3)$$

for all $\xi, \eta \in \mathbb{R}^n$. If in addition the matrix $\mathcal{I}m A$ be symmetric, i.e., $\mathcal{I}m A = \mathcal{I}m A^t$ then this condition is also necessary and is equivalent to

$$|p - 2| |\langle \mathcal{I}m A(x) \xi, \xi \rangle| \leq 2\sqrt{p-1} \langle \mathcal{R}e A(x) \xi, \xi \rangle, \quad \forall \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega. \quad (2.4)$$

(2.4) must hold even in the non-symmetric case, but then this condition might not be sufficient.

To be specific, set

$$\tilde{\mu} = \operatorname{ess\,inf}_{(x,\xi) \in \mathcal{M}} \frac{\langle \mathcal{R}e A(x) \xi, \xi \rangle}{|\langle \mathcal{I}m A(x) \xi, \xi \rangle|} \quad (2.5)$$

where \mathcal{M} is the set of (x, ξ) , $x \in \mathbb{R}^n$, $x \in \Omega$ such that $\langle \mathcal{I}m A(x) \xi, \xi \rangle \neq 0$. If $\mathcal{I}m A = 0$ for any $x \in \Omega$ then \mathcal{L} is L^p dissipative for all $p > 1$. If $\mathcal{I}m A$ is symmetric but does not vanish identically on Ω then \mathcal{L} is L^p dissipative if and only if

$$2 + 2\tilde{\mu} \left(\tilde{\mu} - \sqrt{\tilde{\mu}^2 + 1} \right) \leq p \leq 2 + 2\tilde{\mu} \left(\tilde{\mu} + \sqrt{\tilde{\mu}^2 + 1} \right). \quad (2.6)$$

For our purposes (2.3) is not sufficient as in particular this condition does not imply ellipticity when $p = 2$. To guarantee ellipticity, a stronger lower bound is needed, namely that the left hand side of (2.3) is greater than $\varepsilon(|\frac{2\xi}{\sqrt{pp'}}|^2 + |\eta|^2)$. This yields the condition (1.3), which when $p = 2$ is just the usual ellipticity for complex coefficients.

As we have observed in the introduction, (1.3) can be simply written as $\Delta_p(A) > 0$, where (1.6) defines $\Delta_p(A)$. (This was introduced in [7]) and is in turn equivalent to $|1 - 2/p| < \mu(A)$. Hence the following holds.

Theorem 2.3. *Let $A \in L^\infty(\Omega)$ be a matrix that is uniformly elliptic: for some $\lambda, \Lambda > 0$ and almost every $x \in \Omega$ we have*

$$\lambda |\xi|^2 \leq \mathcal{R}e \sum_{i,j=0}^{n-1} A_{ij}(x) \xi_i \bar{\xi}_j \quad \text{and} \quad |\langle A \xi, \eta \rangle| \leq \Lambda |\xi| |\eta| \quad (2.7)$$

for all $\xi, \eta \in \mathbb{C}^n$. Then there exists $p_0 \in [1, 2)$ (with $p_0 = 1$ if and only if $\mathcal{I}m A = 0$) such that the matrix A is p -elliptic if and only if $p \in (p_0, p'_0)$. That is $\Delta_p(A) > 0$, equivalently

$$\langle \mathcal{R}e A \lambda, \lambda \rangle + \langle \mathcal{R}e A \eta, \eta \rangle + \left\langle \left(\sqrt{\frac{p'}{p}} \mathcal{I}m A - \sqrt{\frac{p}{p'}} \mathcal{I}m A^t \right) \lambda, \eta \right\rangle \geq \varepsilon(p) (|\lambda|^2 + |\eta|^2), \quad (2.8)$$

for some $\varepsilon(p) > 0$ and all $\lambda, \eta \in \mathbb{R}^n$. In fact, $p_0 = \frac{2}{1+\mu(A)}$ where

$$\mu(A) = \operatorname{ess\,inf}_{(x,\xi) \in \mathcal{M}} \mathcal{R}e \frac{\langle A(x) \xi, \xi \rangle}{|\langle A(x) \xi, \bar{\xi} \rangle|} \geq \frac{\lambda}{\Lambda}, \quad (2.9)$$

and \mathcal{M} is the set (x, ξ) with $x \in \Omega$, $\xi \in \mathbb{C}^n$, such that $\langle A(x) \xi, \bar{\xi} \rangle \neq 0$. If in addition the matrix A has symmetric imaginary part ($\mathcal{I}m A = \mathcal{I}m A^t$) this further simplifies to

$$p_0 = 2 + 2\tilde{\mu} \left(\mu - \sqrt{\tilde{\mu}^2 + 1} \right), \quad (2.10)$$

where

$$\tilde{\mu} = \operatorname{ess\,inf}_{(x,\xi) \in \widetilde{\mathcal{M}}} \frac{\langle \mathcal{R}e A(x) \xi, \xi \rangle}{|\langle \mathcal{I}m A(x) \xi, \xi \rangle|} \quad (2.11)$$

and $\widetilde{\mathcal{M}}$ is the set of (x, ξ) with $x \in \Omega$, $\xi \in \mathbb{R}^n$, such that $\langle \mathcal{I}m A(x) \xi, \xi \rangle \neq 0$.

We apply the concept of p -ellipticity to prove the following result.

Theorem 2.4. *Assume that the matrix A is p -elliptic. Then there exists $\lambda'_p = \lambda'_p(\Lambda, \lambda_p) > 0$ such that for any nonnegative, bounded and measurable function χ and any u such that $|u|^{(p-2)/2}u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$, we have*

$$\mathcal{R}e \int_{\Omega} \langle A(x) \nabla u, \nabla(|u|^{p-2}u) \rangle \chi(x) dx \geq \lambda'_p \int_{\Omega} |u|^{p-2} |\nabla u|^2 \chi(x) dx. \quad (2.12)$$

Proof. Since A is p -elliptic (2.8) holds. Changing the variables $\lambda = \frac{2}{\sqrt{pp'}} \xi$ and obtain

$$\begin{aligned} & \frac{4}{pp'} \langle \mathcal{R}e A \xi, \xi \rangle + \langle \mathcal{R}e A \eta, \eta \rangle + \\ & + 2 \langle p^{-1} \mathcal{I}m A - p'^{-1} \mathcal{I}m A^t \rangle \xi, \eta \rangle \geq \varepsilon' (|\xi|^2 + |\eta|^2). \end{aligned} \quad (2.13)$$

Consider now $v \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ and write $|v|^{-1} \bar{v} \nabla v$ as $X + iY$, that is

$$X = \mathcal{R}e(|v|^{-1} \bar{v} \nabla v) \quad \text{and} \quad Y = \mathcal{I}m(|v|^{-1} \bar{v} \nabla v).$$

Let $\chi(x)$ be a nonnegative bounded and measurable function on Ω . By using X in place of ξ and Y in place of η , multiplying by $\chi(x)$ and integrating over Ω one obtains from (2.13) (similar to [8, Corollary 4])

$$\begin{aligned} & \mathcal{R}e \int_{\Omega} \left[\langle A \nabla v, \nabla v \rangle - (1 - 2/p) \langle (A - A^*) \nabla(|v|), |v|^{-1} \bar{v} \nabla v \rangle \right. \\ & \left. - (1 - 2/p)^2 \langle A \nabla(|v|), \nabla(|v|) \rangle \right] \chi(x) dx \geq \varepsilon' \int_{\Omega} |\nabla v|^2 \chi(x) dx \end{aligned} \quad (2.14)$$

Now as in (2.9) of [8] one considers $v = |u|^{p/2-1}u$ and

$$g_{\varepsilon} = (|v|^2 + \varepsilon^2)^{1/2}, \quad u_{\varepsilon} = g_{\varepsilon}^{2/p-1}v,$$

for u, u_{ε} . Using our assumption we see that $|u|^{(p-2)/2}u, |u_{\varepsilon}|^{(p-2)/2}u_{\varepsilon} \in W_{loc}^{1,2}(\Omega, \mathbb{C}^n)$. We let $\varepsilon \rightarrow 0+$. When $p > 2$, we obtain, using Lebesgue dominated convergence,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0+} \mathcal{R}e \int_{\Omega} \langle A \nabla u_{\varepsilon}, \nabla(|u_{\varepsilon}|^{p-2}u_{\varepsilon}) \rangle \chi(x) dx = \\ & \mathcal{R}e \int_{\Omega} \left[\langle A \nabla v, \nabla v \rangle - (1 - 2/p) \langle (A - A^*) \nabla(|v|), |v|^{-1} \bar{v} \nabla v \rangle \right. \\ & \left. - (1 - 2/p)^2 \langle A \nabla(|v|), \nabla(|v|) \rangle \right] \chi(x) dx \geq \varepsilon' \int_{\Omega} |\nabla(|u|^{p/2-1}u)|^2 \chi(x) dx. \end{aligned} \quad (2.15)$$

and then the first term of (2.15) is just

$$\mathcal{R}e \int_{\Omega} \langle A(x) \nabla u, \nabla(|u|^{p-2}u) \rangle \chi(x) dx. \quad (2.16)$$

When $1 < p < 2$, we use a duality argument based on an observation in [8]. Set $w = |u|^{p-2}u$, so that $u = |w|^{p'-2}w$. The fact that $|u|^{(p-2)/2}u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ implies that $|w|^{(p'-2)/2}w \in W_{loc}^{1,2}(\Omega; \mathbb{C})$. Then,

$$\mathcal{R}e \int_{\Omega} \langle A(x) \nabla u, \nabla(|u|^{p-2}u) \rangle \chi(x) dx = \mathcal{R}e \int_{\Omega} \langle A^*(x) \nabla w, \nabla(|w|^{p'-2}w) \rangle \chi(x) dx,$$

and we have that A^* is p' -elliptic when A is p -elliptic. Therefore, we have reduced the regime of the first case considered, and the same limiting argument yields

$$\Re \int_{\Omega} \langle A^*(x) \nabla w, \nabla(|w|^{p'-2} w) \rangle \chi(x) dx \geq \lambda'_{p'} \int_{\Omega} |w|^{p'-2} |\nabla w|^2 \chi(x) dx. \quad (2.17)$$

We conclude the argument for (2.12) in the case $p < 2$ by observing that

$$\int_{\Omega} |w|^{p'-2} |\nabla w|^2 \chi(x) dx \approx \int_{\Omega} |\nabla(|w|^{p'/2-1} w)|^2 \chi(x) dx = \int_{\Omega} |\nabla(|u|^{p/2-1} u)|^2 \chi(x) dx.$$

□

The following observation will be used frequently.

Lemma 2.5. *For all $p > 1$, and for all x for which $u(x) \neq 0$*

$$|\nabla(|u(x)|^{p/2-1} u(x))|^2 \approx |u(x)|^{p-2} |\nabla u(x)|^2.$$

Proof. For $k = 0, 1, \dots, n-1$ we have

$$\partial_k(|u|^{(p-2)/2} u) = |u|^{(p-4)/2} \left[|u| \partial_k u + \frac{p-2}{2} u \partial_k(|u|) \right].$$

Multiplying by its complex conjugate then yields

$$\begin{aligned} \left| \partial_k(|u|^{(p-2)/2} u) \right|^2 &= |u|^{p-4} \left[|u|^2 |\partial_k u|^2 + (p-2) |u| \partial_k(|u|) \Re \langle u, \partial_k u \rangle \right. \\ &\quad \left. + \left(\frac{p-2}{2} \right)^2 |u|^2 (\partial_k(|u|))^2 \right]. \end{aligned}$$

As in the proof of [9, Lemma 2] we have $\partial_k(|u|) = |u|^{-1} \Re \langle u, \partial_k u \rangle$ which yields

$$\left| \partial_k(|u|^{(p-2)/2} u) \right|^2 = |u|^{p-4} \left[|u|^2 |\partial_k u|^2 + \left[(p-2) + \left(\frac{p-2}{2} \right)^2 \right] |\Re \langle u, \partial_k u \rangle|^2 \right].$$

Summing over all k gives us

$$\left| \nabla(|u|^{(p-2)/2} u) \right|^2 = |u|^{p-4} \left[|u|^2 |\nabla u|^2 + \left[\left(\frac{p}{2} \right)^2 - 1 \right] \sum_k |\Re \langle u, \partial_k u \rangle|^2 \right].$$

Since by Cauchy-Schwarz

$$\sum_k |\Re \langle u, \partial_k u \rangle|^2 \leq |u|^2 |\nabla u|^2,$$

then clearly

$$\left| \nabla(|u|^{(p-2)/2} u) \right|^2 \leq \left(1 + \left| \left(\frac{p}{2} \right)^2 - 1 \right| \right) |u|^{p-2} |\nabla u|^2,$$

and for $p > 0$

$$\min \left\{ 1, \left(\frac{p}{2} \right)^2 \right\} |u|^{p-2} |\nabla u|^2 \leq \left| \nabla(|u|^{(p-2)/2} u) \right|^2.$$

Hence the claim holds. □

We now turn to the main lemmas required to prove Theorem 1.1.

Lemma 2.6. *Let the matrix A be p -elliptic for $p \geq 2$ and let B have coefficients satisfying condition (1.9) of Theorem 1.1. Suppose that u is a $W_{loc}^{1,2}(\Omega; \mathbb{C})$ solution to \mathcal{L} in Ω . Then, for any ball $B_r(x)$ with $r < \delta(x)/4$,*

$$\int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \lesssim r^{-2} \int_{B_{2r}(x)} |u(y)|^p dy \quad (2.18)$$

and

$$\left(\int_{B_r(x)} |u(y)|^q dy \right)^{1/q} \lesssim \left(\int_{B_{2r}(x)} |u(y)|^2 dy \right)^{1/2} \quad (2.19)$$

for all $q \in (2, \frac{np}{n-2}]$ when $n > 2$, and where the implied constants depend only p -ellipticity and K of (1.9). When $n = 2$, q can be any number in $(2, \infty)$. In particular, $|u|^{(p-2)/2}u$ belongs to $W_{loc}^{1,2}(\Omega; \mathbb{C})$.

Proof. We begin by assuming that both A and B are smooth. Thus the solution u to $\mathcal{L}(u) = 0$ will be smooth in the interior of Ω ([29]) and we have (2.12) at our disposal for any p such that A is p -elliptic. We prove (2.18) first, and then (2.19) by a bootstrap argument so that no constants appearing depend on the smoothness of the coefficients of A , B .

Let $B_r(x)$ be ball in the interior of Ω with $r < \delta(x)/4$. Let φ be a smooth cutoff function, with $\varphi = 1$ on $B_r(x)$ and vanishing outside of $B_{2r}(x)$. Set $v = u\varphi$. Then,

$$\mathcal{L}v = u\mathcal{L}\varphi + A\nabla u \cdot \nabla \varphi + A^* \nabla u \cdot \nabla \varphi. \quad (2.20)$$

Multiply both sides of (2.20) by $|v|^{p-2}\bar{v}$ and integrate by parts to obtain

$$\begin{aligned} \int \nabla(|v|^{p-2}\bar{v}) \cdot A\nabla v dy &= \int (|v|^{p-2}\bar{v})B \cdot \nabla v dy + \int \nabla(|v|^{p-2}\bar{v}u) \cdot A\nabla \varphi dy \\ &\quad - \int |v|^{p-2}\bar{v}u B \cdot \nabla \varphi dy - \int |v|^{p-2}\bar{v}A\nabla u \cdot \nabla \varphi dy \\ &\quad - \int |v|^{p-2}\bar{v}A^* \nabla u \cdot \nabla \varphi dy \end{aligned} \quad (2.21)$$

The real part of the left hand side of (2.21) is bounded from below by $\lambda_p \int |v|^{p-2} |\nabla v|^2 dy$ by p -ellipticity and (2.12). The first of the five terms on the right hand side above has the bound

$$\left| \int (|v|^{p-2}\bar{v}) \cdot B\nabla v dy \right| \lesssim Kr^{-1} \left(\int |v|^{p-2} |\nabla v|^2 dy \right)^{1/2} \left(\int |v|^p dy \right)^{1/2} \quad (2.22)$$

where K is as in (1.9). The third term has the bound

$$\left| \int |v|^{p-2}\bar{v}u B \cdot \nabla \varphi dy \right| \lesssim Kr^{-2} \int_{B_{2r}(x)} |u|^p dy. \quad (2.23)$$

In terms four and five, we use the fact that $\bar{v}\nabla u = \bar{u}\nabla v - |u|^2\nabla \varphi$ to bound each of these integrals by a constant multiple of

$$\int |v|^{p-2}|u| |\nabla \varphi| |\nabla v| + |v|^{p-2}|u|^2 |\nabla \varphi|^2 dy \quad (2.24)$$

with constant depending on the ellipticity parameter Λ . By Cauchy-Schwarz, this integral is bounded by terms like those in (2.22) and (2.23). After distributing the gradient, the second term has the following bound

$$\int |\nabla(|v|^{p-2}\bar{v})| |u| |\nabla \varphi| + |v|^{p-2} |\bar{v}\nabla u| |\nabla \varphi| dy \quad (2.25)$$

Using Lemma 2.5 to compute the derivative in the first expression, the first term in the above integral is bounded by

$$\int |v|^{p-2} \|\nabla v\| |u| \|\nabla \varphi\| dy \quad (2.26)$$

and the second expression, using the same trick as in (2.24) is bounded by

$$\int |v|^{p-2} \|\nabla v\| |u| \|\nabla \varphi\| + |v|^{p-2} |u|^2 \|\nabla \varphi\|^2 dy \quad (2.27)$$

Taking the real part of both sides of (2.21) and combining all these estimates, and noting that the constants do not depend on the smoothness of the A or B gives (2.18) for this u . Observe, in addition, that the Sobolev embedding gives

$$\left(\int_{B_r(x)} |u|^{\tilde{p}} dy \right)^{1/\tilde{p}} \lesssim \left(\int_{B_{2r}(x)} |v|^{\tilde{p}} dy \right)^{1/\tilde{p}} \lesssim \left(r^2 \int_{B_{2r}(x)} |\nabla(|v|^{p/2-1} v)|^2 dy \right)^{1/p} \quad (2.28)$$

where $\tilde{p} = \frac{pn}{n-2}$. When $n = 2$, this should be interpreted as $\tilde{p} \in (1, \infty)$. In fact, a stronger statement holds, namely that u is Hölder continuous in the interior.

First we use (2.28) and the argument for (2.18) (see (2.21) and the subsequent estimates) to obtain a reverse Hölder inequality for u . That is, for $1 < \alpha \leq 2$,

$$\left(\int_{B_r(x)} |u|^{\tilde{p}} dy \right)^{1/\tilde{p}} \lesssim \left(\int_{B_{\alpha r}(x)} |u|^p dy \right)^{1/p} \quad (2.29)$$

Note that at this moment we have only proven (2.29) for $\alpha = 2$; however, by adjusting the cutoff function φ , the entire argument can be done with balls $B_{\alpha r}(x)$, for any $\alpha > 1$, and with a new constant which will depend on α . Iterating (2.29) k times gives

$$\left(\int_{B_r(x)} |u|^{p_k} dy \right)^{1/p_k} \lesssim \left(\int_{B_{\alpha^k r}(x)} |u|^2 dy \right)^{1/2} \quad (2.30)$$

for $p_k = 2(\frac{n}{n-2})^k$, as long as $p_{k-1} < p$.

It remains to remove the assumption of smoothness of the coefficients of the operator, noting that the constants appearing in (2.18) and (2.19) depend only on the p -ellipticity and K and not on any smoothness parameter. Suppose that A satisfies the condition of p -ellipticity and B satisfies (1.9), and let A_j and B_j be smooth approximations, converging a.e. to A , B , respectively. Let u_j be the solution to $\mathcal{L}_j u_j = 0$ with the same boundary data as u . We claim that the arguments of section 7 of [26] can be used to show that $u_j \rightarrow u$ strongly in $W^{1,2}$ on compact subsets of Ω . We note that the arguments of [26] did not consider the convergence of lower order terms, but this will follow in the same way from the strong convergence of u_j to u in L^q for $q = 2n/(n-2)$. From the fact that (2.19) holds for u_j , and the strong convergence of u_j to u in $L^2(B_{2r})$, we see that the L^q averages of u_j are uniformly bounded. Passing to a subsequence if necessary, we have weak convergence of the u_j in L^q , and that weak limit must be u . This implies that the L^q average of u is bounded as well, i.e., we have (2.19) for u . Now set $w_j = |u_j(x)|^{p/2-1} u_j(x)$ and $w = |u(x)|^{p/2-1} u(x)$. From (2.18) for u_j and (2.19) for u , we have that w_j is uniformly bounded in $W_{\text{loc}}^{1,2}(\Omega; \mathbb{C})$. This gives weak convergence of w_j to a limit, which again must be w . The weak convergence gives the uniform bound on $|\nabla w|^2$, and thus (2.18) for u .

□

Lemma 2.7. *Let the matrix A be p -elliptic for $p < 2$ and let B have coefficients satisfying condition (1.9) of Theorem 1.1. Suppose that u is a $W_{loc}^{1,2}(\Omega; \mathbb{C})$ solution to \mathcal{L} in Ω . Then, for any ball $B_r(x)$ with $r < \delta(x)/4$ and any $\varepsilon > 0$*

$$r^2 \int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \leq C_\varepsilon \int_{B_{2r}(x)} |u(y)|^p dy + \varepsilon \left(\int_{B_{2r}(x)} |u(y)|^2 dy \right)^{p/2} \quad (2.31)$$

and

$$\left(\int_{B_r(x)} |u(y)|^2 dy \right)^{1/2} \leq C_\varepsilon \left(\int_{B_{2r}(x)} |u(y)|^p dy \right)^{1/p} + \varepsilon \left(\int_{B_{2r}(x)} |u(y)|^2 dy \right)^{1/2} \quad (2.32)$$

where the constants depend only p -ellipticity, K of (1.9) and $\varepsilon > 0$. In particular, $|u|^{(p-2)/2}u$ belongs to $W_{loc}^{1,2}(\Omega; \mathbb{C})$.

Proof. We assume that A and B are smooth, and remove this assumption later. We can therefore, as before, assume that we have a smooth solution to \mathcal{L} . The proof of this lemma proceeds in a similar fashion as in (2.19), but without introducing the cutoff function φ . This will lead to presence of the two extra terms (with ε) on the right hand sides of (2.31) and (2.32) which were not needed in (2.18) and (2.19). Unfortunately, if we attempt to proceed as in Lemma 2.6 with $v = u\varphi$ we will run into trouble estimating terms like (2.27) as $|v|^{p-2} \leq |u|^{p-2}$ when $p \geq 2$, but in our case $p < 2$ this does not hold and we run into problems near the boundary of $\text{supp } \varphi$ where φ can be arbitrary small.

The fact that we do not introduce the cutoff function φ does mean we have fewer terms to take care of but we do have to worry about one additional boundary integral and more importantly we also have to introduce an additional approximation to account for the fact that $p - 2$ is negative. To that end, define a cutoff function ρ_δ as follows.

$$\rho_\delta(s) = \begin{cases} \delta^{\frac{p-2}{2}}, & \text{if } 0 \leq s \leq \delta \\ s^{\frac{p-2}{2}}, & \text{if } s > \delta. \end{cases} \quad (2.33)$$

Following the proof of Lemma 2.7, we multiply both sides of the equation $\mathcal{L}u = 0$ by $\rho_\delta(|u|)^2 \bar{u}$ and integrate:

$$\begin{aligned} \int_{B_r(x)} \nabla(\rho_\delta^2(|u|)\bar{u}) \cdot A \nabla u \, dy &= \int_{B_r(x)} (\rho_\delta^2(|u|)\bar{u}) B \cdot \nabla u \, dy \\ &+ \int_{\partial B_r(x)} (\rho_\delta^2(|u|)\bar{u}) \nu \cdot A \nabla u \, d\sigma(y). \end{aligned} \quad (2.34)$$

Here ν is the outer unit normal which for the ball is just $\frac{y-x}{r}$.

The left hand side of (2.34) splits into terms:

$$\begin{aligned} \int_{B_r(x)} \nabla(\rho_\delta^2(|u|)\bar{u}) \cdot A \nabla u \, dy &= \delta^{p-2} \int_{B_r(x) \setminus E_\delta} \langle A \nabla u, \nabla u \rangle \, dy \\ &+ \int_{E_\delta \cap B_r(x)} A \nabla u \cdot \nabla(|u|^{p-2} \bar{u}) \, dy \end{aligned} \quad (2.35)$$

where $E_\delta = \{|u| > \delta\}$. Applying Theorem 2.4 on the open set $E_\delta \cap B_r(x)$, we have that

$$\Re \int_{E_\delta \cap B_r(x)} A \nabla u \cdot \nabla (|u|^{p-2} \bar{u}) dy \geq \lambda_p \int_{E_\delta \cap B_r(x)} |u|^{p-2} |\nabla u|^2 dy \quad (2.36)$$

Ultimately we will let $\delta \rightarrow 0$, and we will show that the integrals involving $B_r(x) \setminus E_\delta$ will tend to zero. The main tool will be the following fact, established in [20] for smooth functions u , namely

$$\delta^r \int_{B_r(x) \setminus E_\delta} |\nabla u|^2 dy \rightarrow 0 \quad (2.37)$$

for all $r > -1$.

We first take care of the boundary integral in (2.34). Observe that (2.34)-(2.36) hold not only on the ball $B_r(x)$ but on any enlarged ball $B_{\alpha r}(x)$ for $1 \leq \alpha \leq 3/2$. Hence if we write (2.34) for each such α and then average over the interval $[1, 3/2]$ the last term of (2.34) will turn into a solid integral over the set $B_{3r/2}(x) \setminus B_r(x)$.

This and (2.35)-(2.36) then yields

$$\begin{aligned} \lambda_p \int_{E_\delta \cap B_r(x)} |u|^{p-2} |\nabla u|^2 dy &\leq \sup_{\alpha \in [1, 3/2]} \left| \int_{B_{\alpha r}(x)} (\rho_\delta^2(|u|) \bar{u}) B \cdot \nabla u dy \right| \\ &+ \left| r^{-1} \int_{B_{3r/2}(x) \setminus B_r(x)} (\rho_\delta^2(|u|) \bar{u}) \frac{(y-x) \cdot A \nabla u}{|y-x|} dy \right| + o(1), \end{aligned} \quad (2.38)$$

where $o(1)$ contains the integral over the complement of E_δ , which tend to zero as $\delta \rightarrow 0$.

Each of the two terms on the right hand side of (2.38) will split into two integrals, one on $B_{3r/2}(x) \setminus E_\delta$ and one on $E_\delta \cap B_{3r/2}(x)$. Clearly,

$$\begin{aligned} \left| \int_{B_{\alpha r}(x)} (\rho_\delta^2(|u|) \bar{u}) B \cdot \nabla u dy \right| &\lesssim r^{-1} \int_{E_\delta \cap B_{3r/2}(x)} |u|^{p-1} |\nabla u| dy \\ &+ \delta^{p-1} r^{-1} \int_{B_{3r/2}(x) \setminus E_\delta} |\nabla u| dy, \end{aligned} \quad (2.39)$$

and

$$\begin{aligned} \left| r^{-1} \int_{B_{3r/2}(x) \setminus B_r(x)} (\rho_\delta^2(|u|) \bar{u}) \frac{(y-x) \cdot A \nabla u}{|y-x|} dy \right| &\lesssim r^{-1} \int_{E_\delta \cap B_{3r/2}(x)} |u|^{p-1} |\nabla u| dy \\ &+ \delta^{p-1} r^{-1} \int_{B_{3r/2}(x) \setminus E_\delta} |\nabla u| dy. \end{aligned} \quad (2.40)$$

where the implied constants in (2.39) and (2.40) depend on K and Λ respectively. The last terms in both inequalities behave like $C\delta^{p-1}$, and will tend to zero as $\delta \rightarrow 0$ and hence can be written as $o(1)$ terms. By Hölder inequality we have for

the first terms on the right hand side of (2.39)-(2.40):

$$\begin{aligned} r^{-1} \int_{E_\delta \cap B_{3r/2}(x)} |u|^{p-1} |\nabla u| dy &\leq r^{-1} \left(\int_{B_{3r/2}(x)} |u|^p dy \right)^{(p-1)/p} \left(\int_{B_{3r/2}(x)} |\nabla u|^p dy \right)^{1/p} \\ &\leq C_\varepsilon r^{-2} \int_{B_{3r/2}(x)} |u|^p dy + \varepsilon r^{p-2} \int_{B_{3r/2}(x)} |\nabla u|^p dy, \end{aligned} \quad (2.41)$$

for any $\varepsilon > 0$.

Putting all terms together therefore yields

$$\lambda_p r^2 \int_{E_\delta} |v|^{p-2} |\nabla v|^2 dy \leq C_\varepsilon \int_{B_{3r/2}(x)} |u|^p dy + \varepsilon r^p \int_{B_{3r/2}(x)} |\nabla u|^p dy + o(1), \quad (2.42)$$

where $o(1)$ contains all integrals over the complement of E_δ , which tend to zero as $\delta \rightarrow 0$. The constant C_ε depends on Λ and K (and ε) but not on the smoothness of A and B . Let δ tend to zero and obtain

$$r^2 \int_{\{y: u(y) \neq 0\} \cap B_r(x)} |u|^{p-2} |\nabla u|^2 dy \leq C'_\varepsilon \int_{B_{2r}(x)} |u|^p dy + \varepsilon r^p \int_{B_{3r/2}(x)} |\nabla u|^p dy. \quad (2.43)$$

Recalling the convention that $|u|^{p-2} |\nabla u|^2$ is taken to be zero whenever $\nabla u = 0$, the integral on the left hand side of (2.43) can be taken on the set $B_r(x) \setminus \{y : u(y) = 0, \nabla u(y) \neq 0\}$. However the measure of $\{y : u(y) = 0, \nabla u(y) \neq 0\}$ is zero and so we can conclude that (after introducing averages)

$$r^2 \oint_{B_r(x)} |u|^{p-2} |\nabla u|^2 dy \leq C''_\varepsilon \oint_{B_{2r}(x)} |u|^p dy + \varepsilon r^p \oint_{B_{3r/2}(x)} |\nabla u|^p dy \quad (2.44)$$

holds. The final step is to use Hölder inequality and Caccioppoli inequality (which is just (2.18) of Lemma 2.6 when $p = 2$) for the last term of (2.44). We have

$$r^p \oint_{B_{3r/2}(x)} |\nabla u|^p dy \lesssim \left(r^2 \oint_{B_{3r/2}(x)} |\nabla u|^2 dy \right)^{p/2} \lesssim \left(\oint_{B_{2r}(x)} |u|^2 dy \right)^{p/2}. \quad (2.45)$$

Finally, (2.44) and (2.45) combined yields (2.31).

The argument for (2.32) is similar to that for the case $p > 2$. First, we have the analog of (2.29).

$$\left(\oint_{B_r(x)} |u|^{\tilde{p}} dy \right)^{1/\tilde{p}} \leq C_\varepsilon \left(\oint_{B_{\alpha r}(x)} |u|^p dy \right)^{1/p} + \varepsilon \left(\oint_{B_{\alpha r}(x)} |u|^2 dy \right)^{1/2} \quad (2.46)$$

The iteration scheme starts with a $p < 2$ and stops when we reach a p_k greater than or equal to 2. Each iteration will give us additional term of the form

$$\left(\oint_{B_{\alpha^i r}(x)} |u|^2 dy \right)^{1/2}$$

on the right hand side for $i = 1, 2, \dots, k$ multiplied by a constant that can be as small as required, since at each step of the iteration ε in (2.46) can be chosen independently of the previous choices.

Finally, since the constants in (2.31) and (2.32) do not depend on the smoothness of the coefficients of A and B , arguments very similar to those in Lemma 2.6 allow

us to pass to the limit and obtain these inequalities for solutions to \mathcal{L} under the assumptions of the Lemma. \square

The reverse Hölder inequalities of Lemmas 2.6 and 2.7 prove Theorem 1.1.

3. CARLESON MEASURES, NONTANGENTIAL MAXIMAL FUNCTIONS AND p -ADAPTED SQUARE FUNCTIONS

3.1. Nontangential maximal and square functions. On a domain of the form

$$\Omega = \{(x_0, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 > \phi(x')\}, \quad (3.1)$$

where $\phi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is a Lipschitz function with Lipschitz constant given by $L := \|\nabla \phi\|_{L^\infty(\mathbb{R}^{n-1})}$, define for each point $x = (x_0, x') \in \Omega$

$$\delta(x) := x_0 - \phi(x') \approx \text{dist}(x, \partial\Omega). \quad (3.2)$$

In other words, $\delta(x)$ is comparable to the distance of the point x from the boundary of Ω .

Definition 3.1. A cone of aperture $a > 0$ is a non-tangential approach region to the point $Q = (x_0, x') \in \partial\Omega$ defined as

$$\Gamma_a(Q) = \{y = (y_0, y') \in \Omega : a|x_0 - y_0| > |x' - y'|\}. \quad (3.3)$$

We require $1/a > L$, otherwise the cone is too large and might not lie inside Ω . But when $\Omega = \mathbb{R}_+^n$ all parameters $a > 0$ may be considered. Sometimes it is necessary to truncate $\Gamma(Q)$ at height h , in which case we write

$$\Gamma_a^h(Q) := \Gamma_a(Q) \cap \{x \in \Omega : \delta(x) \leq h\}. \quad (3.4)$$

Definition 3.2. For $\Omega \subset \mathbb{R}^n$ as above, the square function of some $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by

$$S_a(u)(Q) := \left(\int_{\Gamma_a(Q)} |\nabla u(x)|^2 \delta(x)^{2-n} dx \right)^{1/2} \quad (3.5)$$

and, for each $h > 0$, its truncated version is given by

$$S_a^h(u)(Q) := \left(\int_{\Gamma_a^h(Q)} |\nabla u(x)|^2 \delta(x)^{2-n} dx \right)^{1/2}. \quad (3.6)$$

A simple application of Fubini's theorem gives

$$\|S_a(u)\|_{L^2(\partial\Omega)}^2 \approx \int_{\Omega} |\nabla u(x)|^2 \delta(x) dx. \quad (3.7)$$

In [18], a “ p -adapted” square function was introduced in order to solve Dirichlet problems in the range $1 < p < 2$. We shall use this method, and a similar p -adapted square function, but for both the ranges $p \geq 2$ and $p < 2$. In the following definition, when $p < 2$ we use the convention that the expression $|\nabla u(x)|^2 |u(x)|^{p-2}$ is zero whenever $\nabla u(x)$ vanishes.

Definition 3.3. For $\Omega \subset \mathbb{R}^n$, the p -adapted square function of $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by

$$S_{p,a}(u)(Q) := \left(\int_{\Gamma_a(Q)} |\nabla u(x)|^2 |u(x)|^{p-2} \delta(x)^{2-n} dx \right)^{1/2} \quad (3.8)$$

and, for each $h > 0$, its truncated version is given by

$$S_{p,a}^h(u)(Q) := \left(\int_{\Gamma_a^h(Q)} |\nabla u(x)|^2 |u(x)|^{p-2} \delta(x)^{2-n} dx \right)^{1/2}. \quad (3.9)$$

We have shown (Lemma 2.6) that the expressions of the form $|\nabla u(x)|^2 |u(x)|^{p-2}$, when u is a solution of $\mathcal{L}u = 0$ are locally integrable and hence the definition of $S_p(u)$ makes sense for such p .

Definition 3.4. For $\Omega \subset \mathbb{R}^n$ as above, and for a continuous $u : \Omega \rightarrow \mathbb{C}$, the nontangential maximal function (h -truncated nontangential maximal function) of u at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$, is defined by

$$N_a(u)(Q) := \sup_{x \in \Gamma_a(Q)} |u(x)| \quad \text{and} \quad N_a^h(u)(Q) := \sup_{x \in \Gamma_a^h(Q)} |u(x)|. \quad (3.10)$$

Moreover, we shall also consider a related version of the above nontangential maximal function. This is denoted by $\tilde{N}_{p,a}$ and is defined using L^p averages over balls in the domain Ω . Specifically, given $u \in L_{loc}^p(\Omega; \mathbb{C})$ we set

$$\tilde{N}_{p,a}(u)(Q) := \sup_{x \in \Gamma_a(Q)} w(x) \quad \text{and} \quad \tilde{N}_{p,a}^h(u)(Q) := \sup_{x \in \Gamma_a^h(Q)} w(x) \quad (3.11)$$

for each $Q \in \partial\Omega$ and $h > 0$ where, at each $x \in \Omega$,

$$w(x) := \left(\oint_{B_{\delta(x)/2}(x)} |u(z)|^p dz \right)^{1/p}. \quad (3.12)$$

Above and elsewhere, a barred integral indicates an averaging operation. Observe that, given $u \in L_{loc}^p(\Omega; \mathbb{C})$, the function w associated with u as in (3.12) is continuous and $\tilde{N}_{p,a}(u) = N_a(w)$ everywhere on $\partial\Omega$.

The L^2 -averaged nontangential maximal function was introduced in [26] in connection with the Neuman and regularity problem value problems. In the context of p -ellipticity, Theorem 1.1 shows that there is no difference between L^2 averages and L^p averages, a fact which we record in the following proposition.

Proposition 3.5. Suppose that u is a $W_{loc}^{1,2}(\Omega; \mathbb{C})$ solution to $\mathcal{L} = \text{div}A(x)\nabla + B(x) \cdot \nabla$ in Ω , where the matrix A is assume to be p -elliptic for $p \in (p_0, p'_0)$, and B satisfies condition (1.9). Then, for every $Q \in \partial\Omega$

$$\tilde{N}_{p,a}(u)(Q) \lesssim \tilde{N}_{q,a'}(u)(Q) \quad \forall p, q \in \left(p_0, \frac{p'_0 n}{n-2} \right) \quad (3.13)$$

when the aperture parameters of the cones satisfy $a < a'$ and $p \leq q$ or $q \geq 2$.

When $a < a'$, $q < 2$, $p > q$ and $p, q \in \left(p_0, \frac{p'_0 n}{n-2} \right)$ we have a weaker estimate

$$\tilde{N}_{p,a}(u)(Q) \lesssim \tilde{N}_{q,a'}(u)(Q) + \varepsilon \tilde{N}_{2,a'}(u)(Q), \quad (3.14)$$

for all $\varepsilon > 0$.

Finally, for any aperture parameters a, a' in the appropriate range, and for all p, q in the same range as (3.13),

$$\|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)} \approx \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} \quad (3.15)$$

for all $r > 0$.

Proof. Clearly, (3.13) and (3.14) follows from (1.10) (as we have noted earlier the integral over B_{2r} in (1.10) can be replaced by $B_{\alpha r}$ for any $\alpha > 1$ and hence the cones $\Gamma_{a'}(Q)$ used on the right hand side of (3.13) and (3.14) can be just little bit larger than the cone $\Gamma_a(Q)$ used in definition of $\tilde{N}_{p,a}(u)$ on the left hand side of (3.13) and (3.14)).

Looking at the equivalence in the L^r norm, consider for the moment still the case when $a' > a$. When $p < q$ or $q \geq 2$

$$\|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} \quad (3.16)$$

follows from (3.13). Otherwise by (3.14) applied to $p = 2$ yields

$$\|\tilde{N}_{2,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} + \varepsilon \|\tilde{N}_{2,a'}(u)\|_{L^r(\partial\Omega)}. \quad (3.17)$$

The norm equivalence of the nontangential maximal functions $\tilde{N}_{p,a}$ and $\tilde{N}_{p,a'}$ for different aperture parameters a, a' requires only a classical real-variable argument using the level sets $\{\tilde{N}_{p,a}(u) > \lambda\}$ and $\{\tilde{N}_{p,a'}(u) > \lambda\}$. Hence always

$$\|\tilde{N}_{p,a'}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)}. \quad (3.18)$$

Combining this with (3.17) implies that

$$\|\tilde{N}_{2,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} + \varepsilon \|\tilde{N}_{2,a}(u)\|_{L^r(\partial\Omega)}, \quad (3.19)$$

which by choosing $\varepsilon > 0$ sufficiently small then gives

$$\|\tilde{N}_{2,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)}. \quad (3.20)$$

Combining this with (3.16) for any $p \in \left(p_0, \frac{p_0 n}{n-2}\right)$ and $a < a'' < a'$ we then have

$$\|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{2,a''}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} \quad (3.21)$$

as desired. Hence (3.16) holds for any p, q in our range as long as $a' > a$.

The reverse inequality to (3.16) can be obtained using (3.18) by choosing $a'' < a$. This gives

$$\|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a''}(u)\|_{L^r(\partial\Omega)}$$

and then since $a'' < a$ by (3.16)

$$\|\tilde{N}_{q,a''}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)}.$$

□

3.2. Carleson measures. We begin by recalling the definition of a Carleson measure in a domain Ω as in (3.1). For $P \in \mathbb{R}^n$, define the ball centered at P with the radius $r > 0$ as

$$B_r(P) := \{x \in \mathbb{R}^n : |x - P| < r\}. \quad (3.22)$$

Next, given $Q \in \partial\Omega$, by $\Delta = \Delta_r(Q)$ we denote the surface ball $\partial\Omega \cap B_r(Q)$. The Carleson region $T(\Delta_r)$ is then defined by

$$T(\Delta_r) := \Omega \cap B_r(Q). \quad (3.23)$$

Definition 3.6. A Borel measure μ in Ω is said to be Carleson if there exists a constant $C \in (0, \infty)$ such that for all $Q \in \partial\Omega$ and $r > 0$

$$\mu(T(\Delta_r)) \leq C\sigma(\Delta_r), \quad (3.24)$$

where σ is the surface measure on $\partial\Omega$. The best possible constant C in the above estimate is called the Carleson norm and is denoted by $\|\mu\|_C$.

In all that follows we now assume that the coefficients of the matrix A and B of the elliptic operator $\mathcal{L} = \operatorname{div} A(x) \nabla + B(x) \cdot \nabla$ satisfies the following natural conditions. First, we assume that the entries A_{ij} of A are in $\operatorname{Lip}_{loc}(\Omega)$ and the entries of B are $L_{loc}^\infty(\Omega)$. Second, we assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} [|\nabla A|^2 + |B|^2] \delta(x) dx \quad (3.25)$$

is a Carleson measure in Ω . Sometimes, and for certain coefficients of A , we will assume that their Carleson norm $\|\mu\|_C$ is sufficiently small. Crucially we have the following result.

Theorem 3.7. Suppose that $d\nu = f dx$ and $d\mu(x) = \left[\sup_{B_{\delta(x)/2}(x)} |f| \right] dx$. Assume that μ is a Carleson measure. Then there exists a finite constant $C = C(L, a) > 0$ such that for every $u \in L_{loc}^p(\Omega; \mathbb{C})$ one has

$$\int_{\Omega} |u(x)|^p d\nu(x) \leq C \|\mu\|_C \int_{\partial\Omega} \left(\tilde{N}_{p,a}(u) \right)^p d\sigma. \quad (3.26)$$

Furthermore, consider $\Omega = \mathbb{R}_+^n$ where μ and ν are measures as above supported in Ω and $\delta(x_0, x') = x_0$. Let $h : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^+$ be a Lipschitz function with Lipschitz norm L and

$$\Omega_h = \{(x_0, x') : x_0 > h(x')\}.$$

Then for any $\Delta \subset \mathbb{R}^{n-1}$ with $\sup_{\Delta} h \leq \operatorname{diam}(\Delta)/2$ we have

$$\int_{\Omega_h \cap T(\Delta)} |u(x)|^p d\nu(x) \leq C \|\mu\|_C \int_{\partial\Omega_h \cap T(\Delta)} \left(\tilde{N}_{p,a,h}(u) \right)^p d\sigma. \quad (3.27)$$

Here for a point $Q = (h(x'), x') \in \partial\Omega_h$ we define

$$\tilde{N}_{p,a,h}(u)(Q) = \sup_{\Gamma_a(Q)} w, \quad (3.28)$$

where

$$\Gamma_a(Q) = \Gamma_a((h(x'), x')) = \{y = (y_0, y') \in \Omega : a|h(x') - y_0| > |x' - y'|\} \quad (3.29)$$

and the L^p averages w are defined by (3.12) where the distance δ is taken with respect to the domain $\Omega = \mathbb{R}_+^n$.

Proof. For the first part of the claim let

$$\Omega = \bigcup_i \mathcal{O}_i$$

be a Whitney decomposition of Ω and assume that the Whitney sets \mathcal{O}_i are such that for any $x \in \mathcal{O}_i$ we have $\mathcal{O}_i \subset B_{\delta(x)/2}(x)$. Also, $|\mathcal{O}_i| \approx |B_{\delta(x)/2}(x)|$. It follows that on each \mathcal{O}_i we have

$$\int_{\mathcal{O}_i} |u(x)|^p d\nu(x) \leq \left[\sup_{\mathcal{O}_i} |f| \right] \int_{\mathcal{O}_i} |u(x)|^p dx.$$

By the definition (3.12) for w it follows that for any $y \in \mathcal{O}_i$ we have

$$\int_{\mathcal{O}_i} |u(x)|^p d\nu(x) \lesssim \left[\sup_{B_{\delta(y)/2}(y)} |f| \right] w(y)^p |\mathcal{O}_i|.$$

From this

$$\int_{\mathcal{O}_i} |u(x)|^p d\nu(x) \lesssim \int_{\mathcal{O}_i} w(y)^p d\mu(y). \quad (3.30)$$

Summing over all i we get

$$\int_{\Omega} |u(x)|^p d\nu(x) \lesssim \int_{\Omega} w(y)^p d\mu(y) \lesssim \|\mu\|_C \int_{\partial\Omega} N_a(w)^p d\sigma,$$

where the last inequality follows from the usual inequality for the Carleson measure. Since $\tilde{N}_{p,a}(u) = N_a(w)$ the claim follows.

The second claim has a similar argument. Because h is Lipschitz and $\sup_{\Delta} h \leq \text{diam}(\Delta)/2$ it follows that

$$\Omega_h \cap T(\Delta) = \bigcup_i \mathcal{O}_i$$

where again \mathcal{O}_i are Whitney sets with respect to the original domain $\Omega = \mathbb{R}_+^n$. It follows that for each i once again (3.30) holds. Summing over i we get

$$\int_{\Omega_h \cap T(\Delta)} |u(x)|^p d\nu(x) \lesssim \int_{\Omega_h \cap T(\Delta)} w(y)^p d\mu(y) \lesssim \|\mu\|_C \int_{\partial\Omega_h \cap T(\Delta)} N_{a,h}(w)^p d\sigma,$$

where the last inequality is a standard estimate for a Carleson measure on a Lipschitz domain. \square

3.3. Pullback Transformation. The Carleson measure conditions, (3.25), on the coefficients of \mathcal{L} are compatible with a useful change of variables described in this subsection.

For a domain Ω as in (3.1), consider a mapping $\rho : \mathbb{R}_+^n \rightarrow \Omega$ appearing in works of Dahlberg, Nečas, Kenig-Stein and others, defined by

$$\rho(x_0, x') := (x_0 + P_{\gamma x_0} * \phi(x'), x'), \quad \forall (x_0, x') \in \mathbb{R}_+^n, \quad (3.31)$$

for some positive constant γ . Here P is a nonnegative function $P \in C_0^\infty(\mathbb{R}^{n-1})$ and, for each $\lambda > 0$,

$$P_\lambda(x') := \lambda^{-n+1} P(x'/\lambda), \quad \forall x' \in \mathbb{R}^{n-1}. \quad (3.32)$$

Finally, $P_\lambda * \phi(x')$ is the convolution

$$P_\lambda * \phi(x') := \int_{\mathbb{R}^{n-1}} P_\lambda(x' - y') \phi(y') dy'. \quad (3.33)$$

Observe that ρ extends up to the boundary of \mathbb{R}_+^n and maps one-to-one from $\partial\mathbb{R}_+^n$ onto $\partial\Omega$. Also for sufficiently small $\gamma \lesssim L$ the map ρ is a bijection from $\overline{\mathbb{R}_+^n}$ onto $\overline{\Omega}$ and, hence, invertible.

For $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ that solves $\mathcal{L}u = 0$ in Ω with Dirichlet datum f consider $v := u \circ \rho$ and $\tilde{f} := f \circ \rho$. The change of variables via the map ρ just described implies that $v \in W_{loc}^{1,2}(\mathbb{R}_+^n; \mathbb{C})$ solves a new elliptic PDE of the form

$$0 = \text{div}(\tilde{A}(x)\nabla v) + \tilde{B}(x) \cdot \nabla v, \quad (3.34)$$

with boundary datum \tilde{f} on $\partial\mathbb{R}_+^n$. Hence, solving a boundary value problem for u in Ω is equivalent to solving a related boundary value problem for v in \mathbb{R}_+^n . Crucially,

if the coefficients of the original system are such that (3.25) is a Carleson measure, then the coefficients of \tilde{A} and \tilde{B} satisfy an analogous Carleson condition in the upper-half space. If, in addition, the Carleson norm of (3.25) is small and L (the Lipschitz constant for the domain Ω) is also small, then the Carleson norm for the new coefficients \tilde{A} and \tilde{B} will be correspondingly small. Hence the map ρ allows us to assume that the domain is $\Omega = \mathbb{R}_+^n$.

Moreover, this transformation also preserves p -ellipticity.

4. THE L^p -DIRICHLET PROBLEM

When an operator \mathcal{L} is as in Theorem 1.2 is uniformly elliptic in the sense of (1.1), the Lax-Milgram lemma can be applied and guarantees the existence of weak solutions. That is, given any $f \in \dot{B}_{1/2}^{2,2}(\partial\Omega; \mathbb{C})$, the homogenous space of traces of functions in $\dot{W}^{1,2}(\Omega; \mathbb{C})$, there exists a unique (up to a constant) $u \in \dot{W}^{1,2}(\Omega; \mathbb{C})$ such that $\mathcal{L}u = 0$ in Ω and $\text{Tr } u = f$ on $\partial\Omega$. We call these solutions “energy solutions” and use them to define the notion of solvability of the L^p Dirichlet problem.

Definition 4.1. *Let Ω be the Lipschitz domain introduced in (3.1) and fix an integrability exponent $p \in (1, \infty)$. Also, fix an aperture parameter $a > 0$. Consider the following Dirichlet problem for a complex valued function $u : \Omega \rightarrow \mathbb{C}$:*

$$\begin{cases} 0 = \partial_i (A_{ij}(x) \partial_j u) + B_i(x) \partial_i u & \text{in } \Omega, \\ u(x) = f(x) & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\ \tilde{N}_{2,a}(u) \in L^p(\partial\Omega), \end{cases} \quad (4.1)$$

where the usual Einstein summation convention over repeated indices (i, j in this case) is employed.

We say the Dirichlet problem (4.1) is solvable for a given $p \in (1, \infty)$ if there exists a $C = C(p, \Omega) > 0$ such that for all boundary data $f \in L^p(\partial\Omega; \mathbb{C}) \cap \dot{B}_{1/2}^{2,2}(\partial\Omega; \mathbb{C})$ the unique “energy solution” satisfies the estimate

$$\|\tilde{N}_{2,a}(u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{C})}. \quad (4.2)$$

Remark. Given $f \in \dot{B}_{1/2}^{2,2}(\partial\Omega; \mathbb{C}) \cap L^p(\partial\Omega; \mathbb{C})$ the corresponding energy solution constructed above is unique (since the decay of L^p eliminates constant solutions). As the space $\dot{B}_{1/2}^{2,2}(\partial\Omega; \mathbb{C}) \cap L^p(\partial\Omega; \mathbb{C})$ is dense in $L^p(\partial\Omega; \mathbb{C})$ for each $p \in (1, \infty)$, it follows that there exists a unique continuous extension of the solution operator $f \mapsto u$ to the whole space $L^p(\partial\Omega; \mathbb{C})$, with u such that $\tilde{N}_{2,a}(u) \in L^p(\partial\Omega)$ and the accompanying estimate $\|\tilde{N}_{2,a}(u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{C})}$ being valid.

Moreover, we shall establish in the appendix that under the assumptions of Theorem 1.2 for any $f \in L^p(\partial\Omega; \mathbb{C})$ the corresponding solution u constructed by the continuous extension attains the datum f as its boundary values in the following sense. Consider the average $\tilde{u} : \Omega \rightarrow \mathbb{C}$ defined by

$$\tilde{u}(x) = \int_{B_{\delta(x)/2}(x)} u(y) dy, \quad \forall x \in \Omega.$$

Then

$$f(Q) = \lim_{x \rightarrow Q, x \in \Gamma(Q)} \tilde{u}(x), \quad \text{for a.e. } Q \in \partial\Omega, \quad (4.3)$$

where the a.e. convergence is taken with respect to the \mathcal{H}^{n-1} Hausdorff measure on $\partial\Omega$.

Let us make some observations that explain the structural assumptions we have made in Theorem 1.2. As we have already stated it suffices to formulate the result in the case $\Omega = \mathbb{R}_+^n$ by using the pull-back map introduced above. Since Theorem 1.2 requires that the coefficients have *small* Carleson norm this puts a restriction on the size of the Lipschitz constant $L = \|\nabla\phi\|_{L^\infty}$ of the map ϕ that defines the domain Ω in (3.1). The constant L will have also to be small (depending on λ_p , Λ , n and p).

For technical reasons in the proof we also need that all coefficients A_{0j} , $j = 0, 1, \dots, n-1$ are real. This can be ensured as follows. When $j > 0$ observe that we have

$$\partial_0([\mathcal{I}m A_{0j}]\partial_j u) = \partial_j([\mathcal{I}m A_{0j}]\partial_0 u) + (\partial_0[\mathcal{I}m A_{0j}])\partial_j u - ([\partial_j \mathcal{I}m A_{0j}])\partial_0 u \quad (4.4)$$

which allows to move the imaginary part of the coefficient A_{0j} onto the coefficient A_{j0} at the expense of two (harmless) first order terms. This does not work for the coefficient A_{00} . Instead we make the following observation.

Suppose that the measure (3.25) associated to an operator $\mathcal{L} = \partial_i(A_{ij}(x)\partial_j) + B_i(x)\partial_i$ is Carleson. Consider a related operator $\tilde{\mathcal{L}} = \partial_i(\tilde{A}_{ij}(x)\partial_j) + \tilde{B}_i(x)\partial_i$, where $\tilde{A} = \alpha A$ and $\tilde{B} = \alpha B - (\partial_i \alpha)A_{ij}$, and $\alpha \in L^\infty(\Omega)$ is a complex valued function such that $|\alpha(x)| \geq \alpha_0 > 0$ and $|\nabla \alpha|^2 x_0$ is a Carleson measure.

Observe that a weak solution u to $\tilde{\mathcal{L}}u = 0$ is also a weak solution to $\mathcal{L}u = 0$ and that the new coefficients of \tilde{A} and \tilde{B} also satisfy a Carleson measure condition as in (3.25), from the assumption on α . We will only require that the coefficient \tilde{A}_{00} is real but we may as well ensure for simplicity that it equals to 1. Clearly, if we choose $\alpha = A_{00}^{-1}$, then the new operator $\tilde{\mathcal{L}}$ will have this property. When A_{00} (and hence α) is real, then so is \tilde{A}_{00} . Similarly, if A is p -elliptic and $\mathcal{I}m A_{00}$ is sufficiently small (depending on the ellipticity constants), then \tilde{A} will also be p -elliptic. However, if $\mathcal{I}m \alpha$ is not small, the p -ellipticity, after multiplication of A by α may not be preserved. Thus, we assume in our main result (Theorem 1.2) the p -ellipticity of the new matrix \tilde{A} which has all coefficients \tilde{A}_{0j} , $j = 0, 1, \dots, n-1$ real, as this is not implied in the general case from the p -ellipticity of the original matrix A .

The solutions to the Dirichlet problem in the infinite domain \mathbb{R}_+^n will be obtained as a limit of solutions in infinite strips $\Omega^h = \{x = (x_0, x')\} \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < x_0 < h\}$. We define them as follows.

Definition 4.2. Let $\Omega = \mathbb{R}_+^n$, and let Ω^h be the infinite strip

$$\Omega^h = \{x = (x_0, x')\} \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < x_0 < h\},$$

and let $p \in (1, \infty)$. Also, fix an aperture parameter $a > 0$. Let u be a complex valued function $u : \Omega \rightarrow \mathbb{C}$ such that

$$\begin{cases} 0 = \partial_i (A_{ij}(x) \partial_j u) + B_i(x) \partial_i u & \text{in } \Omega^h, \\ u(x_0, x') = 0, & \text{for all } x_0 \geq h, \\ u(x) = f(x) & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\ \tilde{N}_{2,a}(u) \in L^p(\partial\Omega), \end{cases} \quad (4.5)$$

where the usual Einstein summation convention over repeated indices (i, j in this case) is employed.

We say the Dirichlet problem (4.5) is solvable for a given $p \in (1, \infty)$ if there exists a $C = C(p, \Omega) > 0$ such that for all boundary data $f \in L^p(\partial\Omega; \mathbb{C}) \cap B_{1/2}^{2,2}(\partial\Omega; \mathbb{C})$ we have that $u|_{\Omega^h}$ is the unique “energy solution” to

$$\begin{cases} 0 = \partial_i (A_{ij}(x) \partial_j u) + B_i(x) \partial_i u & \text{in } \Omega^h, \\ u(x_0, x') = 0, & \text{for } x_0 = h \\ u(x) = f(x) & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \end{cases} \quad (4.6)$$

and satisfies the estimate

$$\|\tilde{N}_{2,a}(u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{C})}. \quad (4.7)$$

We now give the proof of Theorem 1.2. We will establish the solvability of the Dirichlet problem (4.5) assuming that the coefficients of A and B are smooth, applying the results of sections 5 and 6. The constants will not depend on the degree of smoothness or on the width of the strip. Then, a limiting argument proves Theorem 1.2 for smooth coefficients. Finally, we consider smooth approximations of \mathcal{L} , and another limiting argument gives Theorem 1.2.

Proof. Let u_h be the energy solution in Ω^h as in Definition 4.2. As follows from Corollary 5.2

$$\lambda'_p \iint_{\mathbb{R}_+^n} |\nabla u_h|^2 |u_h|^{p-2} x_0 dx' dx_0 \leq \int_{\mathbb{R}^{n-1}} |f(x')|^p dx' + C \|\mu'\|_C \int_{\mathbb{R}^{n-1}} [\tilde{N}_{p,a}(u_h)]^p dx'. \quad (4.8)$$

We shall momentarily assume finiteness of the quantities involving nontangential maximal functions and square functions needed to apply Corollary 6.2, and return to this point later. Thus we have

$$\int_{\mathbb{R}^{n-1}} [\tilde{N}_{p,a}(u_h)]^p dx' \leq C_1 \int_{\mathbb{R}^{n-1}} |f(x')|^p dx' + C_2 \|\mu'\|_C \int_{\mathbb{R}^{n-1}} [\tilde{N}_{p,a}(u_h)]^p dx'. \quad (4.9)$$

Here the constants C_1, C_2 depend on p, λ_p, Λ, n and $\|\mu\|_C$. It follows that for

$$\|\mu'\|_C < \frac{1}{2C_2}$$

we have that

$$\int_{\mathbb{R}^{n-1}} [\tilde{N}_{p,a}(u_h)]^p dx' \leq 2C_1 \int_{\mathbb{R}^{n-1}} |f(x')|^p dx'. \quad (4.10)$$

We now consider the limit of u_h , as $h \rightarrow \infty$. The uniform Lax-Milgram estimate on $\|\nabla u_h\|_{L^2(\mathbb{R}_+^n)}$ by $\|f\|_{\dot{B}_{1/2}^{2,2}}$, and the fact that $\text{Tr}(u_h) = f$, gives a weakly convergent subsequence to some u with $\|\nabla u\|_{L^2(\mathbb{R}_+^n)} \leq C \|f\|_{\dot{B}_{1/2}^{2,2}}$ and $\text{Tr}(u) = f$. This

subsequence is therefore strongly convergent to u in $L^2_{loc}(\mathbb{R}^n_+)$. It follows that the L^2 averages w_h of u_h converge locally and uniformly to w , the L^2 averages of u in $C_{loc}(\mathbb{R}^n_+)$.

Let $\Gamma_k(x')$ be the doubly truncated cone $\Gamma(x') \cap \{1/k < x_0 < k\}$. Define

$$\tilde{N}_k(u)(x') = \sup_{y \in \Gamma_k(x')} |w(y)|,$$

and with $\tilde{N}_k(u_h)(x')$ defined analogously. Then we have

$$\tilde{N}_k(u_h)(x') \rightarrow \tilde{N}_k(u)(x') \quad \text{uniformly on compact subsets } K \subset \mathbb{R}^{n-1}.$$

Finally, using (4.10), this give on each such set K ,

$$\|\tilde{N}_k(u)\|_{L^p(K)} = \lim_{h \rightarrow \infty} \|\tilde{N}_k(u_h)\|_{L^p(K)} \leq C\|f\|_{L^p(\mathbb{R}^{n-1})}.$$

The constant C in the estimate above is independent of K and k , so that taking the supremum in each of k and K gives the desired estimate assuming the coefficients are smooth. Finally, we approximate our coefficients by smooth functions, and the passage from the smooth coefficient case requires a further argument that mirrors the limiting process above using truncations of cones.

We now address the finiteness requirements of Corollary 6.2, in two separate cases. In the a priori estimate $\|\tilde{N}_{p,a}(u_h)\|_{L^p(\mathbb{R}^{n-1})} < c$, the constant c is allowed to depend on measures of smoothness of the coefficients or on the truncation parameter h , as the only property used to obtain estimate (4.10) is the fact that $\|\tilde{N}_{p,a}(u_h)\|_{L^p(\mathbb{R}^{n-1})}$ is finite.

Consider first the case $p \geq 2$. First,

$$\|\tilde{N}_{p,a}(u_h)\|_{L^2(\mathbb{R}^{n-1})}^2 \lesssim \|S_{2,a}(u_h)\|_{L^2(\mathbb{R}^{n-1})}^2 \lesssim \int_{\Omega_h} |\nabla u_h|^2 dx < \infty.$$

By interpolation with the Agmon-Douglis-Nirenberg ([5]) L^∞ bound for solutions to smooth systems, it follows that $\|\tilde{N}_{p,a}(u_h)\|_{L^p(\mathbb{R}^{n-1})} < \infty$. Since $p \geq p'$, this suffices to apply Corollary 6.2.

Now suppose that $p < 2$. In this case, we need that both $\|\tilde{N}_{p,a}(u_h)\|_{L^p(\mathbb{R}^{n-1})} < \infty$ and $\|S_{p',a}(u_h)\|_{L^{p'}(\mathbb{R}^{n-1})}^p < \infty$. We shall use an extrapolation argument based on a method in [13] of obtaining $L^{2-\varepsilon}$ estimates of nontangential maximal functions from L^2 estimates on sawtooth domains. See also [17], where this technique was used to get solvability of the L^p Dirichlet problem for elliptic systems for $2 - \varepsilon < p < 2$. In particular, the argument of [13], reproduced in section 6 of [17] for systems and hence valid in our setting, gives that $\|\tilde{N}_{2,a}(u_h)\|_{L^{p_0}(\mathbb{R}^{n-1})} < \infty$ for $p_0 = 2 - \varepsilon$ and hence the same is true for $\|\tilde{N}_{p_0,a}(u_h)\|_{L^{p_0}(\mathbb{R}^{n-1})}$. The quantity ε depends on the constant C_2 in the L^2 norm inequality between the nontangential maximal function and the square function S_2 . We now observe that this gives $\|S_{p'_0,a}(u_h)\|_{L^{p_0}(\mathbb{R}^{n-1})}^2 < \infty$ as well: Use the fact that, pointwise,

$$S_{p'_0,a}(u_h) < C_\eta S_{p_0,a}(u_h) + \eta N(u_h)$$

where N is the pointwise maximal function, and η is as small as we wish, together with Corollary 5.2 to bound $\|S_{p_0,a}(u_h)\|_{L^{p_0}(\mathbb{R}^{n-1})}$ by $\|\tilde{N}_{p_0,a}(u_h)\|_{L^{p_0}(\mathbb{R}^{n-1})}$.

Once these two quantities are finite, Corollary 6.2 applies and we obtain (4.10), and hence the same estimate for u , for $p_0 = 2 - \varepsilon$ and a constant $C_{2-\varepsilon}$.

The very same argument, now invoking the L^{p_0} estimate gives an $L^{p_0-\varepsilon'}$ estimate where ε' now depends on $C_{2-\varepsilon}$. In other words, we apply the same argument as

[13] but starting from known estimates for the nontangential maximal function in L^{p_0} instead of L^2 . There is no difference in the structure of the argument. We can continue this bootstrapping as long as we stay in the range of p -ellipticity and as long as we can be sure that we are moving by an amount ε which is not getting smaller at each step. This last point is assured by the fact that the constants C_p in the L^p norm inequalities (4.10) only depend on the Lipschitz constants, p -ellipticity and the Carleson measure norm of the coefficients. Thus, for fixed $q < 2$ where the operator is q -elliptic, the constants C_p for $q < p < 2$ are uniformly bounded. \square

5. ESTIMATES FOR THE p -ADAPTED SQUARE FUNCTION $S_p(u)$

In this, and the next, section we make the assumption that the coefficients of A and B are smooth, in order to ensure the finiteness of L^p norms of the nontangential maximal function.

We fix an $h > 1$, and an infinite strip Ω^h defined above, and let u be an energy solution to (4.6), extended to be zero above height h . In this section we establish a one sided estimate of the p -adapted square function of u in terms of boundary data and its nontangential maximal function, with constants independent of h .

Lemma 5.1. *Let $u : \Omega \rightarrow \mathbb{C}$ be as above, with the Dirichlet boundary datum $f \in \dot{B}_{1/2}^{2,2}(\partial\Omega; \mathbb{C}) \cap L^p(\partial\Omega; \mathbb{C})$. Assume that A is p -elliptic and smooth in \mathbb{R}_+^n with $A_{00} = 1$ and A_{0j} real and that the measure μ' defined as in (1.13) is Carleson. Then there exists a constant $C = C(\lambda_p, \Lambda, p, n)$ such that for all $r > 0$*

$$\begin{aligned} p \frac{\lambda_p}{2} \iint_{[0, r/2] \times \partial\Omega} |u|^{p-2} |\nabla u|^2 x_0 dx' dx_0 + \frac{2}{r} \iint_{[0, r] \times \partial\Omega} |u(x_0, x')|^p dx' dx_0 \\ \leq \int_{\partial\Omega} |f(x')|^p dx' + \int_{\partial\Omega} |u(r, x')|^p dx' + C \|\mu'\|_C \int_{\partial\Omega} \left[\tilde{N}_{p,a}^r(u) \right]^p dx'. \end{aligned} \quad (5.1)$$

Proof. To proceed, fix an arbitrary $y' \in \partial\Omega \equiv \mathbb{R}^{n-1}$, and consider first an $r \leq h$. Pick a smooth cutoff function ζ which is x_0 -independent and satisfies

$$\zeta = \begin{cases} 1 & \text{in } B_r(y'), \\ 0 & \text{outside } B_{2r}(y'). \end{cases} \quad (5.2)$$

Moreover, assume that $r|\nabla\zeta| \leq c$ for some positive constant c independent of y' . We begin by considering the integral quantity

$$\mathcal{I} := \mathcal{R}e \iint_{[0, r] \times B_{2r}(y')} A_{ij} \partial_j u \partial_i (|u|^{p-2} \bar{u}) x_0 \zeta dx' dx_0 \quad (5.3)$$

with the usual summation convention understood. With $\chi = x_0 \zeta$ we have by Theorem 2.4 for some $\lambda_p > 0$

$$\mathcal{I} \geq \lambda_p \iint_{[0, r] \times B_{2r}} |u|^{p-2} |\nabla u|^2 x_0 \zeta dx' dx_0, \quad (5.4)$$

where we agree henceforth to abbreviate $B_{2r} := B_{2r}(y')$ whenever convenient.

The idea now is to integrate by parts the formula for \mathcal{I} in order to relocate the ∂_i derivative. This gives

$$\begin{aligned}
\mathcal{I} &= \mathcal{R}e \int_{\partial[(0,r) \times B_{2r}]} A_{ij} \partial_j u |u|^{p-2} \bar{u} x_0 \zeta \nu_{x_i} d\sigma \\
&\quad - \mathcal{R}e \iint_{[0,r] \times B_{2r}} \partial_i (A_{ij} \partial_j u) |u|^{p-2} \bar{u} x_0 \zeta dx' dx_0 \\
&\quad - \mathcal{R}e \iint_{[0,r] \times B_{2r}} A_{ij} \partial_j u |u|^{p-2} \bar{u} \partial_i x_0 \zeta dx' dx_0 \\
&\quad - \mathcal{R}e \iint_{[0,r] \times B_{2r}} A_{ij} \partial_j u |u|^{p-2} \bar{u} x_0 \partial_i \zeta dx' dx_0 \\
&=: I + II + III + IV,
\end{aligned} \tag{5.5}$$

where ν is the outer unit normal vector to $(0, r) \times B_{2r}$. The boundary term I is zero except on the set $\{r\} \times B_{2r}$ and only when $i = 0$. This gives

$$I = \mathcal{R}e \int_{\{r\} \times B_{2r}} A_{0j} \partial_j u |u|^{p-2} \bar{u} x_0 \zeta d\sigma \tag{5.6}$$

As u is a weak solution of $\mathcal{L}u = 0$ in Ω , we use the equation to transform II into

$$II = \mathcal{R}e \iint_{[0,r] \times B_{2r}} B_i (\partial_i u) |u|^{p-2} \bar{u} x_0 \zeta dx' dx_0. \tag{5.7}$$

To further estimate this term we use Hölder's inequality, the Carleson condition for the term B and Theorem 3.7 in order to write

$$\begin{aligned}
|II| &\leq \left(\iint_{[0,r] \times B_{2r}} |B|^2 |u|^p x_0 \zeta dx' dx_0 \right)^{1/2} \cdot \left(\iint_{[0,r] \times B_{2r}} |u|^{p-2} |\nabla u|^2 x_0 \zeta dx' dx_0 \right)^{1/2} \\
&\leq C(\lambda_p, \Lambda, p, n) \left(\|\mu'\|_C \int_{B_{2r}} \left[\tilde{N}_{p,a}^r(u) \right]^p dx' \right)^{1/2} \cdot \mathcal{I}^{1/2}.
\end{aligned} \tag{5.8}$$

As $\partial_i x_0 = 0$ for $i > 0$ the term III is non-vanishing only for $i = 0$. We further split this term by considering the cases when $j = 0$ and $j > 0$. This yields, since $A_{00} = 1$,

$$\begin{aligned}
III_{\{j=0\}} &= -\mathcal{R}e \iint_{[0,r] \times B_{2r}} \partial_0 u |u|^{p-2} \bar{u} \zeta dx' dx_0 \\
&= -\frac{1}{p} \iint_{[0,r] \times B_{2r}} \partial_0 (|u|^p) \zeta dx' dx_0 \\
&= -\frac{1}{p} \int_{B_{2r}} |u|^p(r, x') \zeta dx' + \frac{1}{p} \int_{B_{2r}} |u|^p(0, x') \zeta dx'
\end{aligned} \tag{5.9}$$

When $j > 0$ we first use the fact that A_{0j} is real and hence the expression $\mathcal{R}e [A_{0j} (\partial_j u) |u|^{p-2} \bar{u}] = p^{-1} A_{0j} \partial_j (|u|^p)$. Then we reintroduce $1 = \partial_0 x_0$ and integrate by parts moving the ∂_0 derivative

$$\begin{aligned}
III_{\{j \neq 0\}} &= -\mathcal{R}e \iint_{[0,r] \times B_{2r}} A_{0j} \partial_j u |u|^{p-2} \bar{u} \zeta \, dx' \, dx_0 \\
&\quad - p^{-1} \iint_{[0,r] \times B_{2r}} A_{0j} \partial_j (|u|^p) (\partial_0 x_0) \zeta \, dx' \, dx_0 \\
&= p^{-1} \int_{B_{2r}} A_{0j} \partial_j (|u|^p)(r, x') r \zeta \, dx' + p^{-1} \iint_{[0,r] \times B_{2r}} \partial_0 A_{0j} \partial_j (|u|^p) x_0 \zeta \, dx' \, dx_0 \\
&\quad + p^{-1} \iint_{[0,r] \times B_{2r}} A_{0j} \partial_{0j}^2 (|u|^p) x_0 \zeta \, dx' \, dx_0 \\
&= III_1 + III_2 + III_3.
\end{aligned}$$

We note that $III_1 = -I_{\{j \neq 0\}}$.

In the third term III_3 we switch the order of derivatives $\partial_{0j}^2 = \partial_{j0}^2$ and take further integration by parts with respect to ∂_j .

$$\begin{aligned}
III_3 &= -p^{-1} \iint_{[0,r] \times B_{2r}} \partial_j A_{0j} \partial_0 (|u|^p) x_0 \zeta \, dx' \, dx_0 \\
&\quad - p^{-1} \iint_{[0,r] \times B_{2r}} A_{0j} \partial_0 (|u|^p) x_0 (\partial_j \zeta) \, dx' \, dx_0 = III_{31} + III_{32}.
\end{aligned}$$

The terms III_2 and III_{31} are of the same type as II we have handled earlier and hence have the same estimate

$$III_2 + III_{31} \leq C(\lambda_p, \Lambda, p, n) \left(\|\mu'\|_C \int_{B_{2r}} \left[\tilde{N}_{p,a}^r(u) \right]^p \, dx' \right)^{1/2} \cdot \mathcal{I}^{1/2}$$

We add up all terms we have so far to obtain

$$\begin{aligned}
\mathcal{I} &\leq p^{-1} \int_{B_{2r}} \partial_0 (|u|^p)(r, x') r \zeta \, dx' \\
&\quad - p^{-1} \int_{B_{2r}} |u|^p(r, x') \zeta \, dx' + p^{-1} \int_{B_{2r}} |u|^p(0, x') \zeta \, dx' \\
&\quad + C(\lambda_p, \Lambda, p, n) \|\mu'\|_C \int_{B_{2r}} \left[\tilde{N}_{p,a}^r(u) \right]^p (u) \, dx' + \frac{1}{2} \mathcal{I} \\
&\quad + III_{32} + IV.
\end{aligned} \tag{5.10}$$

We have used the arithmetic-geometric inequality for expression bounding the term II in (5.8) as well as for similar terms III_2 and III_{31} .

To obtain a global version of (5.10), consider a sequence of disjoint boundary balls $(B_r(y'_k))_{k \in \mathbb{N}}$ such that $\cup_k B_{2r}(y'_k)$ covers $\partial\Omega = \mathbb{R}^{n-1}$ and consider a partition of unity $(\zeta_k)_{k \in \mathbb{N}}$ subordinate to this cover. That is, assume $\sum_k \zeta_k = 1$ on \mathbb{R}^{n-1} and each ζ_k is supported in $B_{2r}(y'_k)$. Write IV_k for each term as the last expression in (5.5) corresponding to $B_{2r} = B_{2r}(y'_k)$. Given that $\sum_k \partial_i \zeta_k = 0$ for each i , by summing (5.10) over all k 's gives $\sum_k IV_k = 0$. The same observation applies to the

terms arising in III_{32} . It follows that

$$\begin{aligned} \frac{\lambda_p}{2} \iint_{[0,r] \times \mathbb{R}^{n-1}} |\nabla u|^2 |u|^{p-2} x_0 dx' dx_0 &\leq \\ p^{-1} \int_{\mathbb{R}^{n-1}} \partial_0(|u|^p)(r, x') r dx' & \\ - p^{-1} \int_{\mathbb{R}^{n-1}} |u|^p(r, x') dx' + p^{-1} \int_{\mathbb{R}^{n-1}} |u|^p(0, x') dx' & \\ + C \|\mu'\|_C \int_{\mathbb{R}^{n-1}} \left[\tilde{N}_{p,a}^r(u) \right]^p dx'. & \end{aligned} \quad (5.11)$$

We have established (5.11) for $r \leq h$, but we now observe that (5.11) holds also for $r > h$, as $u = 0$ when $r \geq h$. To see this, note that when $r = h$, the second term on the right hand side of the inequality is negative, and the third term is zero. From this, (5.1) follows by integrating (5.11) in r on $[0, r']$ and dividing by r' . \square

Lemma 5.1, and its proof, yields several important corollaries.

Corollary 5.2. *Under the assumptions of Lemma 5.1 we have for such u :*

$$\lambda'_p \iint_{\mathbb{R}_+^n} |\nabla u|^2 |u|^{p-2} x_0 dx' dx_0 \leq \int_{\mathbb{R}^{n-1}} |f(x')|^p dx' + C \|\mu'\|_C \int_{\mathbb{R}^{n-1}} \left[\tilde{N}_{p,a}(u) \right]^p dx'. \quad (5.12)$$

Furthermore, under the same assumptions, if $g : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^+$ is a Lipschitz function with small Lipschitz norm for any $\Delta \subset \mathbb{R}^{n-1}$ such that $\sup_{\Delta} g \leq d/2$ where $d = \text{diam}(\Delta)$ we also have the following local estimate

$$\iint_{\Omega_g \cap T(\Delta)} |\nabla u|^2 |u|^{p-2} \delta_g(x) dx \leq C \int_{2\Delta} \left(|u(g(x'), x')|^p + (1 + \|\mu\|_C) \left[\tilde{N}_{p,a,g}^{2d}(u) \right]^p \right) dx'. \quad (5.13)$$

Here $\tilde{N}_{p,a,g}^{2d}$ is the truncated version of the nontangential maximal function defined in (3.28) with respect to the domain $\Omega_g = \{x_0 > g(x')\}$ and δ_g measures the distance of a point to the boundary of Ω_g .

Proof. The first claim follows immediately from Lemma 5.1 by taking $r \rightarrow \infty$ since $\int_{\partial\Omega} |u(r, x')|^p dx' = 0$ when $r > h$.

The second claim can be seen as follows. In the case when the function $g(x') = 0$, one proceeds exactly as in the proof of Lemma 5.1 above until (5.10). Then instead of summing over different balls $\Delta = B_r$ covering \mathbb{R}^{n-1} we estimate the terms III_{32} and IV . Both of these terms are of the same type and can be bounded (up to a constant) by

$$\iint_{[0,r] \times B_{2r}} |\nabla u| |u|^{p-1} x_0 |\partial_T \zeta| dx' dx_0, \quad (5.14)$$

where $\partial_T \zeta$ denotes any of the derivatives in the direction parallel to the boundary. Recall that ζ is a smooth cutoff function equal to 1 on B_r and 0 outside B_{2r} . In particular, we may assume ζ to be of the form $\zeta = \eta^2$ for another smooth function η such that $|\nabla_T \eta| \leq C/r$. By Cauchy-Schwarz (5.14) can be further estimated by

$$\left(\iint_{[0,r] \times B_{2r}} |\nabla u|^2 |u|^{p-2} x_0 (\eta)^2 dx' dx_0 \right)^{1/2} \left(\iint_{[0,r] \times B_{2r}} |u|^p x_0 |\nabla_T \eta|^2 dx' dx_0 \right)^{1/2} \quad (5.15)$$

$$\lesssim \mathcal{I}^{1/2} \left(\frac{1}{r} \iint_{[0,r] \times B_{2r}} |u|^p dx' dx_0 \right)^{1/2} \leq \varepsilon \mathcal{I} + C_\varepsilon \int_{B_{2r}} \left[\tilde{N}_{p,a,g}^r(u) \right]^p dx'.$$

In the last step we have used the AG-inequality and a straightforward estimate of the solid integral $|u|^p$ by the p -averaged nontangential maximal function. Substituting (5.15) into (5.10) the estimate (5.13) follows by integrating in r over $[0, r']$ and dividing by r' exactly as done above. We note that, by the second part of Theorem 3.7, we can use in the estimates $\tilde{N}_{p,a}$ defined as in (3.28).

In the general case, for g Lipschitz with small constant, we use the pullback map to again work on \mathbb{R}_+^n . Recall that we working in the infinite strip Ω^h , but this will transform under this mapping. Let $G(x')$ be the image of the line $x_0 = h$ under this pullback. Instead of integrating on $[0, r]$ in Lemma 5.1, we integrate first in x' and then on $[0, r'(x')]$, where $r' = \min(r, G(x'))$. Since u vanishes at $(x', G(x'))$, there will still be no contribution from boundary integrals when integrating derivatives $\partial_j u$, $j < n-1$, and the argument goes through as before. \square

Lemma 5.3. *Let $\Omega = \mathbb{R}_+^n$ and assume u be the energy solution of (4.6) Assume that A is p -elliptic and smooth in \mathbb{R}_+^n with $A_{00} = 1$ and A_{0j} real and that the measure μ defined as in (1.12) is Carleson.*

Consider any $b > a > 0$. Then for each $\gamma \in (0, 1)$ there exists a constant $C(\gamma) > 0$ such that $C(\gamma, a, b) \rightarrow 0$ as $\gamma \rightarrow 0$ and with the property that for each $\nu > 0$ and each energy solution u of (4.1) there holds

$$\left| \left\{ x' \in \mathbb{R}^{n-1} : S_{p,a}(u)(x') > \nu, \tilde{N}_b(u)(x') \leq \gamma \nu \right\} \right| \leq C(\gamma) \left| \left\{ x' \in \mathbb{R}^{n-1} : S_{p,b}(u)(x') > \nu/2 \right\} \right|. \quad (5.16)$$

Proof. We observe that $\{x' \in \mathbb{R}^{n-1} : S_{p,b}(u)(x') > \nu/2\}$ is an open subset of \mathbb{R}^{n-1} . When this set is empty, or is all of \mathbb{R}^{n-1} , estimate (5.16) is trivial, so we focus on the case when the set in question is both nonempty and proper. Granted this, we may consider a Whitney decomposition $(\Delta_i)_{i \in I}$ of it, consisting of open cubes in \mathbb{R}^{n-1} . Let F_ν^i be the set appearing on the left-hand side of (5.16) intersected with Δ_i . Let r_i be the diameter of Δ_i . Due to the nature of the Whitney decomposition there exists a point $p' \in 2\Delta_i$ such that $S_{p,b}(u)(p') < \nu/2$. From this and the fact that $b > a$ it follows that for all $x' \in F_\nu^i$ we have

$$S_{p,a}^d(u)(x') > \nu/2,$$

where $S_{p,a}^d$ is the truncated version of the square function at some height $d \approx r_i$, where the precise nature of relation between d and r_i depends on the apertures a and b .

For some $a < c < b$ consider the domain

$$\Omega_c = \bigcup_{x' \in F_\nu^i} \Gamma_c(x');$$

this is a Lipschitz domain with Lipschitz constant $1/c$. Observe that $F_\nu^i \subset \partial\Omega_c$. It follows that

$$|F_\nu^i| \leq \frac{2^p}{\nu^p} \int_{F_\nu^i} [S_{p,a}^d(u)(x')]^p dx' \lesssim \nu^{-p} \iint_{\Omega_c \cap T(\Delta_i)} |\nabla u|^2 |u|^{p-2} \delta_c dx,$$

where δ_c measures the distance of a point to the boundary $\partial\Omega_c$. It follow by (5.13)

$$|F_\nu^i| \lesssim \nu^{-p} \int_{\partial\Omega_c \cap T(2\Delta_i)} \left(|u|_{\partial\Omega_c} \right)^p + (1 + \|\mu\|_c) \left[\tilde{N}_{p,a,c}^{2d}(u) \right]^p d\sigma,$$

where $\tilde{N}_{p,a,c}^{2d}$ is defined using nontangential cones with aperture a with vertices on $\partial\Omega_c$. Due to the fact that each of these cones is contained in one of the cones $\Gamma_b(x')$ for some $x' \in F_\nu^i$ (as $c < b$) and on F_ν^i : $\tilde{N}_b(u)(x') \leq \gamma\nu$ we also have $\tilde{N}_{p,a,c}^{2d}(u) \leq \gamma\nu$ everywhere on $\partial\Omega_c$. Thus we can conclude

$$|F_\nu^i| \lesssim \nu^{-p} \int_{\partial\Omega_c \cap T(2\Delta_i)} |u|_{\partial\Omega_c}^p d\sigma + (1 + \|\mu\|_c) \nu^{-p} (\gamma\nu)^p |2\Delta_i|.$$

We still need to deal with the first term on the righthand side. We convert this term into a solid integral by averaging c over the interval $[a, b]$. Such solid integral has a trivial estimate by

$$C \int_{\partial\Omega_b \cap T(2\Delta_i)} \left[\tilde{N}_{p,a,b}^{2d}(u) \right]^p d\sigma \lesssim (\gamma\nu)^p |2\Delta_i|.$$

Hence using that the surface measure is doubling we finally get

$$|F_\nu^i| \leq C(a, b, \|\mu\|_c) \gamma^p |\Delta_i|.$$

Summing over all i yields (5.16) as desired. \square

We will require a localized version of Lemma 5.3 in order to deal with integrals of square functions and nontangential maximal functions that are not a priori finite.

Lemma 5.4. *Let u , defined with respect to Ω^h , and A, a, b , be as in Lemma 5.3. Fix $R \leq h$ and consider the boundary ball $\Delta_R \subset \mathbb{R}^{n-1}$. Let $p > q > 1$ for any q such that A is q -elliptic. Let*

$$\nu_0^p = C \int_{\Delta_{2R}} N_b^p(u) dx',$$

where C is a constant depending only on dimension (calculated in the proof below). Then for each $\gamma \in (0, 1)$ there exists a constant $C(\gamma) > 0$ such that $C(\gamma, a, b) \rightarrow 0$ as $\gamma \rightarrow 0$ and with the property that for each $\nu > \nu_0$

$$\begin{aligned} & \left| \left\{ x' \in \Delta_R : S_{q,a}(u)(x') > \nu, \tilde{N}_b(u)(x') \leq \gamma\nu \right\} \right| \\ & \leq C(\gamma) \left| \left\{ x' \in \Delta_R : S_{q,b}(u)(x') > \nu/2 \right\} \right|. \end{aligned} \quad (5.17)$$

Proof. By Corollary 5.2, $\|S_{q,b}(u)\|_{L^q(\Delta_R)} \lesssim \|N_b(u)\|_{L^q(\Delta_{2R})}$. Therefore,

$$|\Delta_R \cap \{S_q > \nu/2\}| \lesssim \nu^{-q} \|N_b(u)\|_{L^q(\Delta_{2R})}^q \quad (5.18)$$

$$\begin{aligned} & \lesssim \nu^{-q} \|N_b(u)\|_{L^p(\Delta_{2R})}^{q/p} |\Delta_{2R}|^{1-q/p} \\ & \lesssim C_\varepsilon \nu^{-p} \int_{\Delta_{2R}} (N_b(u))^p + \varepsilon |\Delta_R|. \end{aligned} \quad (5.19)$$

Choosing $\varepsilon = 1/4$, which determines C_ε , and we now fix $C = 4C_\varepsilon$ in the definition of ν_0 . This implies that for any $\nu > \nu_0$, we have that

$$|\Delta_R \cap \{S_{q,b} > \nu/2\}| < 1/2 |\Delta_R|.$$

Thus, there exists a Whitney decomposition of $\Delta_R \cap \{S_{q,b} > \nu/2\}$ into open cubes Δ_i with the property that $2\Delta_i \cap \Delta_R$ contains a point for which $S_{q,b}(u) < \nu/2$. From this point on, the proof proceeds as in Lemma 5.3. \square

Corollary 5.5. *Under the assumption of Lemma 5.3, for any $q \geq p > 1$ and $a > 0$ there exists a finite constant $C = C(\lambda_p, \Lambda, p, q, a, \|\mu\|_C, n) > 0$ such that*

$$\|S_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})} \leq C \|\tilde{N}_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})}. \quad (5.20)$$

The statement also holds for any $q > 0$, provided we know a priori that $\|S_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})} < \infty$.

Proof. This is a consequence of the good- λ inequality established above and the equivalence ([11]) of p -adapted square functions with different aperture in any L^q norm.

When $q \geq p$, and M is large,

$$\int_0^M \nu^{q-1} |\Delta_R \cap \{S_{p,a}(u) > \nu\}| d\nu \leq C(M) \int_0^M \nu^{p-1} |\Delta_R \cap \{S_{p,a}(u) > \nu\}| d\nu.$$

By Corollary 5.2, and the fact that the coefficients are smooth, the right hand side is bounded. Therefore, the left hand side is also bounded, with a constant that may depend on M .

Now we multiply the good- λ inequality of Lemma 5.4 by ν^{p-1} and integrate separately over $(0, \nu_0)$ and (ν_0, M) . This gives

$$\|S_{p,a}(u)\|_{L^q(\Delta_R)} \leq C \|\tilde{N}_{p,a}(u)\|_{L^q(\Delta_{2R})},$$

after taking the limit as $M \rightarrow \infty$.

The estimate (5.20) follows after summing over a decomposition of \mathbb{R}^{n-1} into balls of size R and adding the local estimates.

When $q < p$, the local good- λ inequality is not available, which is why we need the additional a priori estimate on the finiteness of $\|S_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})}$. \square

6. BOUNDS FOR THE NONTANGENTIAL MAXIMAL FUNCTION BY THE p -ADAPTED SQUARE FUNCTION

As before, we work on $\Omega = \mathbb{R}_+^n$ and we assume that the matrix A is p -elliptic. Our aim in this section is to establish the converse of the inequality in Corollary 5.5. The approach necessarily differs from the usual argument in the real scalar elliptic case due to the fact that certain estimates, such as interior Hölder regularity of a weak solution, are unavailable for the complex coefficient case. Hence, alternative arguments bypassing such difficulties must be devised. We use here an adaptation of the approach developed for elliptic systems in [17].

Since any scalar complex valued PDE can be written as a real skew-symmetric system, the theorem of [17] can be applied here and we have the following result (c.f. Proposition 5.8 of [17]).

Proposition 6.1. *Let u be an arbitrary energy solution of (4.1) in $\Omega = \mathbb{R}_+^n$. Assume that A is elliptic and the measure μ defined as in (1.12) is Carleson with*

norm $\|\mu\|_{\mathcal{C}} < \infty$. Then for any $q > 0$ and $a > 0$ there exists a finite constant $C = C(\lambda, \Lambda, q, a, \|\mu\|_{\mathcal{C}}, n) > 0$ such that

$$\|\tilde{N}_{2,a}(u)\|_{L^q(\mathbb{R}^{n-1})} \leq C \|S_{2,a}(u)\|_{L^q(\mathbb{R}^{n-1})}. \quad (6.1)$$

Corollary 6.2. Let $\Omega = \mathbb{R}_+^n$ and assume u is a solution of (4.6). Assume that A is p -elliptic and smooth in \mathbb{R}_+^n with $A_{00} = 1$ and A_{0j} real and that the measure μ defined as in (1.12) is Carleson. Then for any $q > 0$ and $a > 0$ there exists a finite constant $C = C(\lambda_p, \Lambda, p, q, a, \|\mu\|_{\mathcal{C}}, n) > 0$ such that

$$\|\tilde{N}_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})} \leq C \|S_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})}, \quad (6.2)$$

provided that a priori $\|\tilde{N}_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})} < \infty$. If the dual exponent $p' > q$ we also have to assume that $\|S_{p',a}(u)\|_{L^q(\mathbb{R}^{n-1})} < \infty$.

Proof. Using Hölder's inequality we have for any $x' \in \mathbb{R}^{n-1}$

$$\begin{aligned} [S_{2,a}(u)(x')]^2 &= \iint_{\Gamma_a(x')} |\nabla u|^{2/p} |u|^{1-2/p} |\nabla u|^{2/p'} |u|^{1-2/p'} x_0 dx' dx_0 \\ &\leq \left(\iint_{\Gamma_a(x')} |\nabla u|^2 |u|^{p-2} x_0 dx' dx_0 \right)^{1/p} \left(\iint_{\Gamma_a(x')} |\nabla u|^2 |u|^{p'-2} x_0 dx' dx_0 \right)^{1/p'} \\ &\leq S_{p,a}(u)(x') S_{p',a}(u)(x'). \end{aligned} \quad (6.3)$$

It follows by Proposition 6.1

$$\|\tilde{N}_{2,a}(u)\|_{L^q(\mathbb{R}^{n-1})}^2 \leq C \|S_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})} \|S_{p',a}(u)\|_{L^q(\mathbb{R}^{n-1})}. \quad (6.4)$$

A p -elliptic matrix A is also p' -elliptic and hence Corollary 5.5 applies. This gives

$$\|S_{p',a}(u)\|_{L^q(\mathbb{R}^{n-1})} \leq C \|\tilde{N}_{p',a}(u)\|_{L^q(\mathbb{R}^{n-1})}. \quad (6.5)$$

Combining these two estimates with Proposition 3.5 we have

$$\begin{aligned} \|\tilde{N}_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})}^2 &\lesssim \|\tilde{N}_{2,a}(u)\|_{L^q(\mathbb{R}^{n-1})}^2 \\ &\lesssim \|S_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})} \|\tilde{N}_{p',a}(u)\|_{L^q(\mathbb{R}^{n-1})} \\ &\lesssim \|S_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})} \|\tilde{N}_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})}. \end{aligned} \quad (6.6)$$

From this our claim follows. \square

7. APPENDIX: BOUNDARY VALUES OF SOLUTIONS WITH $\|\tilde{N}_{2,a}(u)\|_{L^p} < \infty$.

The results in this section are of a general nature, and have applications to issues of nontangential convergence of solutions in the boundary value problems considered in this paper.

We start by considering an energy solution $u \in \dot{W}^{1,2}(\Omega; \mathbb{C})$ of the Dirichlet boundary value problem (4.1). Denote by $\tilde{u} : \Omega \rightarrow \mathbb{C}$ the averages

$$\tilde{u}(x) = \oint_{B_{\delta(x)/2}(x)} u(y) dy, \quad \forall x \in \Omega.$$

Clearly, \tilde{u} is a continuous function on Ω . We shall establish the following lemma.

Lemma 7.1. *For $u \in \dot{W}^{1,2}(\Omega; \mathbb{C})$ let $f = \text{Tr } u$ be its trace on $\partial\Omega$ (which belongs to the space $\dot{B}_{1/2}^{2,2}(\partial\Omega; \mathbb{C})$). Then*

$$f(Q) = \lim_{x \rightarrow Q, x \in \Gamma(Q)} \tilde{u}(x), \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } Q \in \partial\Omega. \quad (7.1)$$

Proof. It suffices to work on $\Omega = \mathbb{R}_+^n$ since the pull-back transformation (3.31) defines a bijection between $\dot{W}^{1,2}(\Omega; \mathbb{C})$ and $\dot{W}^{1,2}(\mathbb{R}_+^n; \mathbb{C})$ and maps an interior ball $B_{\delta(x)/2}(x)$ in Ω into an open set on \mathbb{R}_+^n that contains and is contained in balls of radius comparable to $\delta(x)/2$. Hence the result proven on \mathbb{R}_+^n transfers to Ω .

Hence from now on let $\Omega = \mathbb{R}_+^n$. Writing $x \in \mathbb{R}_+^n$ as (x_0, x') consider the functions

$$\tilde{u}_k(x') = \tilde{u}(2^{-k}, x'), \quad \forall x' \in \mathbb{R}^{n-1}.$$

Then for any $x, y \in \mathbb{R}_+^n$ with $|x - y| \leq r$ and $\delta(x), \delta(y) \approx r$ we have

$$|\tilde{u}(x) - \tilde{u}(y)|^2 \lesssim \int_{\mathcal{H}} |\nabla u|^2 r^{2-n} dy \quad (7.2)$$

where \mathcal{H} is the convex hull of the set $B_{\delta(x)/2}(x) \cup B_{\delta(y)/2}(y)$. It follows that

$$\int_{\mathbb{R}^{n-1}} |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')|^2 dx' \lesssim \int_{(2^{-(k-1)}, 2^{-(k+2)}) \times \mathbb{R}^{n-1}} |\nabla u|^2 (2^{-k}) dy \leq 2^{-k} \|\nabla u\|_{L^2(\mathbb{R}_+^n)}^2.$$

From this we have that $(\tilde{u}_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $L_{loc}^2(\mathbb{R}^{n-1})$ and hence convergent. As f is the trace of u it follows that $\tilde{u}_k \rightarrow f$ in L_{loc}^2 .

Next we show that $\tilde{u}_k \rightarrow f$ pointwise almost everywhere. For any $\lambda > 0$ consider the set

$$E_\lambda = \left\{ x' \in \mathbb{R}^{n-1} : \forall k \in \mathbb{N} \text{ we have } |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')|^2 \leq \frac{\lambda}{2^{k/2}} \right\}.$$

We estimate the size of the complement of E_λ . Clearly,

$$\begin{aligned} |E_\lambda^c| &\leq \sum_{k=1}^{\infty} \left| \left\{ x' \in \mathbb{R}^{n-1} : |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')|^2 > \frac{\lambda}{2^{k/2}} \right\} \right| \\ &\leq \sum_{k=1}^{\infty} \frac{2^{k/2}}{\lambda} \int_{\mathbb{R}^{n-1}} |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')|^2 dx' \leq \sum_{k=1}^{\infty} \frac{2^{k/2}}{\lambda} 2^{-k} \|\nabla u\|_{L^2(\mathbb{R}_+^n)}^2 \leq \frac{C}{\lambda}. \end{aligned}$$

It follows that $\cap_{\lambda>0} E_\lambda^c$ is a set of measure zero. Hence the set

$$\mathcal{S} = \left(\bigcup_{\lambda>0} E_\lambda \right) \cap \{x' \in \mathbb{R}^{n-1} : S_{10a}^1(u)(x') < \infty\}$$

has full measure. Here $S_{10a}^1(u)$ is the truncated square function at the height 1. Clearly, $\{x' \in \mathbb{R}^{n-1} : S_{10a}^1(u)(x') < \infty\}$ is a set of full measure due to our assumption that $u \in W^{1,2}(\mathbb{R}_+^n)$. Indeed,

$$\int_{\mathbb{R}^{n-1}} S_{10a}^1(u)(x') dx' \approx \int_{\mathbb{R}^{n-1} \times (0,1)} |\nabla u|^2 x_0 dx_0 dx' \leq \|\nabla u\|_{L^2(\mathbb{R}_+^n)}^2 < \infty,$$

and hence $S_{10a}^1(u)(x') < \infty$ a.e. as claimed and therefore \mathcal{S} is a set of full measure.

Consider any $y \in \Gamma_a^1(x')$ for $x' \in \mathcal{S}$. Find an integer $k \in \mathbb{N}$ such that $\delta(y) \approx 2^{-k}$ and hence also $|y - (2^{-k}, x')| \approx 2^{-k}$. We estimate the difference $\tilde{u}(y) - \tilde{u}_k(x')$. As before we have (c.f. (7.2))

$$|\tilde{u}(y) - \tilde{u}_k(x')|^2 \lesssim \int_{\mathcal{H}} |\nabla u(y)|^2 \delta(y)^{2-n} dy \lesssim \int_{\mathcal{O}_{k-2} \cup \mathcal{O}_{k-1} \cup \mathcal{O}_k \cup \mathcal{O}_{k+1}} |\nabla u(y)|^2 \delta(y)^{2-n} dy$$

where \mathcal{H} is the convex hull of the set $B_{\delta((2^{-k}, x'))/2}((2^{-k}, x')) \cup B_{\delta(y)/2}(y)$. Here

$$\mathcal{O}_j = \{(y_0, y') \in \Gamma_{10a}(x') : y_0 \in (2^{-j}, 2^{-j+1}]\}.$$

Since

$$[S_{10a}^1(u)(x')]^2 = \sum_{k=1}^{\infty} \int_{\mathcal{O}_k} |\nabla u(y)|^2 \delta(y)^{2-n} dy < \infty$$

we see that

$$\int_{\mathcal{O}_k} |\nabla u(y)|^2 \delta(y)^{2-n} dy \rightarrow 0, \quad \text{as } k \rightarrow \infty$$

and hence

$$|\tilde{u}(y) - \tilde{u}_k(x')| \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (7.3)$$

Consider now the sequence $(\tilde{u}_k(x'))_{k \in \mathbb{N}}$. We claim that it is Cauchy and hence convergent. Indeed, since $x' \in \mathcal{S}$ then $x' \in E_\lambda$ for some $\lambda > 0$ and hence

$$\sum_{k=1}^{\infty} |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')| \leq \sum_{k=1}^{\infty} \frac{\sqrt{\lambda}}{2^{k/4}} < \infty.$$

From this the claim that $(\tilde{u}_k(x'))_{k \in \mathbb{N}}$ is Cauchy follows. As $\tilde{u}_k \rightarrow f$ in L_{loc}^2 we therefore have $\tilde{u}_k(x') \rightarrow f(x')$ pointwise as $k \rightarrow \infty$ for all $x' \in \mathcal{S}$ (f can be modified on a set of measure zero if necessary). Combining this with (7.3) we see that

$$\tilde{u}(y) \rightarrow f(x'), \quad \text{as } y \rightarrow x' \text{ and } y \in \Gamma_a(x')$$

for all $x' \in \mathcal{S}$. This proves the lemma. \square

Lemma 7.2. *Let $1 < p < \infty$ and assume that the L^p Dirichlet problem for the operator $\mathcal{L}u = \operatorname{div}(A(x)\nabla u) + B(x) \cdot \nabla u$ is solvable on a domain $\Omega \subset \mathbb{R}^n$. Assume also that \mathcal{L} is such that the Lax-Milgram lemma applies (implying existence of the energy solutions in $\dot{W}^{1,2}(\Omega; \mathbb{C})$).*

For any $f \in L^p(\partial\Omega; \mathbb{C})$ consider an approximation of f by functions $f_k \in \dot{B}_{1/2}^{2,2}(\partial\Omega; \mathbb{C}) \cap L^p(\partial\Omega; \mathbb{C})$ such that

$$f_k \rightarrow f \quad \text{in } L^p(\partial\Omega; \mathbb{C}).$$

Let u_k be the energy solutions corresponding to data given by f_k . Let

$$u = \lim_{k \rightarrow \infty} u_k \quad \text{on } \Omega.$$

Then $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ and u satisfies the estimate

$$\|\tilde{N}_{2,a}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)} \quad (7.4)$$

with $C > 0$ as in Definition 4.1. The averages

$$\tilde{u}(x) = \oint_{B_{\delta(x)/2}(x)} u(y) dy, \quad \forall x \in \Omega$$

satisfy

$$f(Q) = \lim_{y \rightarrow Q, y \in \Gamma(Q)} \tilde{u}(y), \quad \text{for a.e. } Q \in \partial\Omega. \quad (7.5)$$

We omit the proof of the lemma as it uses the same argument as in the case of real coefficients, repeatedly using the estimate that follows from solvability:

$$\|\tilde{N}_{2,a}(u_k - u_l)\|_{L^p(\partial\Omega)} \leq C\|f_k - f_l\|_{L^p(\partial\Omega)}.$$

In the real case the approximating functions are chosen so that they are continuous, which under mild assumptions on the regularity of $\partial\Omega$ (such as NTA) then immediately implies

$$f_k(Q) = \lim_{y \rightarrow Q, y \in \Gamma(Q)} \tilde{u}_k(y), \quad \text{for all } Q \in \partial\Omega. \quad (7.6)$$

In our case (of complex coefficients) (7.6) is replaced by (7.1) for each u_k and f_k (the a.e. convergence is sufficient for the argument). The rest of the proof goes as in the real case giving us (7.5) for \tilde{u} and f .

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SCHOOL OF MATHEMATICS,, THE UNIVERSITY OF EDINBURGH AND MAXWELL INSTITUTE OF MATHEMATICAL SCIENCES, UK

E-mail address: M.Dindos@ed.ac.uk

DEPARTMENT OF MATHEMATICS,, BROWN UNIVERSITY, USA

E-mail address: jill.pipher@brown.edu